Notes 4.2 Name: ____________________

Factor Analysis

Theorem. *a* divides *b* if for each prime *p*, the exponent of *p* in *a* is less than that in *b*.

Example 1. 6 is a factor of 120 because 6 * 20 = 120, or because 6 = 2 * 3 and 120 = $2^3 * 3 * 5$, and each exponent of a prime in 6 is less than or equal to the corresponding exponent of the prime in 120.

Let $n = p_1^{e_1} p_2^{e_2} .. p_k^{e_k}$ be the prime factorization of n.

Number of Factors: $(e_1 + 1)(e_2 + 1)(e_3 + 1)\ldots(e_k + 1)$.
Proof. Clearly a factor must be of the form $p_1^{x_1} p_2^{x_2} .. p_k^{x_k}$, where $0 \le x_i \le e_i$. Thus, each x_i can be one of $e_i + 1$ values, implying the formula by constructive counting.

Product of Factors: We take factors in pairs (if x is a factor then so is n/x), and so the product of factors of n is $n^{(\#\ factors\ of\ n)/2}$.

Sum of Factors: Observe this is given by
$(1 + p_1 + p_1^2 + p_1^3 + \ldots + p_1^{e_1})(1 + p_2 + p_2^2 + p_2^3 + \ldots + p_2^{e_2})\ldots(1 + p_k + p_k^2 + p_k^3 + \ldots + p_k^{e_k})$,
because on multiplying, each factor of n is represented exactly once.

Exercise 1. What is the number of square factors of $2^{200} 3^{199} 5^{198}$?

Example 2. What is the sum of the squares of factors of 144?
Solution. 144 = $2^4 3^2$. Consider $(1 + 2^2 + 2^4 + 2^6 + 2^8)(1 + 3^2 + 3^4)$. If we multiply this out, each square of a factor of 144 is represented exactly once in the sum. Thus, our answer is 341 * 91 = 31031.

Exercise 2. What is the sum of the cubes of factors of 72?

Exercise 3. Use the geometric series formula to write the sum of factors formula in a different way.

Exercise 41. What is the prime factorization of the product of the factors of (a) 72 (b) 144 (c) 112 (d) 91 (e) 1000 (f) 160?

Exercise 42. Find the sum of the sum of the factors of 36

Exercise 43. Define a *perfect number* to be a number equal to the sum of its proper factors (factors not including itself). For example, 6 is perfect because 1 + 2 + 3 = 6. Define an *abundant number* to be a number *n* whose sum of proper factors is greater than *n*; define a *deficient number* to be a number *n* whose sum of proper factors is less than *n*.

a. Show that 28 is a perfect number.
b. Determine whether 7, 13, 26, 18, 30, 25 are perfect, abundant, or deficient.
c. Prove that all prime numbers are deficient.
d. Show that all numbers of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is prime, are perfect numbers.