PTMS MATH LEAGUE

PROJECT NOTES

A GUIDE TO COMPETITIVE MIDDLE SCHOOL MATHEMATICS

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Notes 0.0
Common Sense

It is a PRIVILEGE to attend math competitions, which means you earn the right and you can also lose the right due to improper behavior.

Here are some things to keep in mind.

1. **Solve the problems.** Each competition will have a set of problems to solve. Some of them are easier than others. But when you get to harder questions you can't solve, don't give up and walk out of the room. Rather, try persevering on the problem and maybe you'll even find the solution. Also, checking your work might help you avoid silly mistakes on previous problems.
2. **Don't disturb others.** Even when others are disturbing you, don't respond. Save the comments for after the test.
3. **Be polite.** After the test, there will be opportunities to discuss answers and solutions. Sometimes you breathe a sigh of relief because you got the answer correct, but other times you might have a wrong answer. In any case, you must keep calm: don't shout with joy or wail in despair.
4. **Follow directions.** When a problem asks for a simplified fraction, don't put a mixed number as the answer. Also remember to put your name on the test.
5. **Respect others.** Not everyone is as smart as you are, but they are attending the competition for a prestigious reason. Being condescending and arrogant does not help anyone. This is different from being competitive; it is possible to be both competitive and respectful. For example, it is still possible to shake the adversary's hands even after losing to him or her in Countdown round.
6. **Don't bring contraband.** If calculators are not allowed, don't bring them to the test. You should also check the rules for when rulers and compasses are allowed. Failure to comply with the rules may result in disqualification.
7. **Be fluent at English.** Just because English is not your best subject doesn't mean it is not important. All problems require you to read several lines of text; misinterpreting them is one of the easiest ways to get the problem wrong.
8. **Be mature.** It is one thing to be funny; it is another to be rude and inappropriate. In a civilized society, maturity acts as the lubricant to keep social interactions polished; a lack thereof will cause disorder and on an individual level might disqualify people from the competition (this actually happened at ARML once).

CHAPTER ONE

GEOMETRY

Geometry Theorem Sheet Name: ___________________

Area/Perimeter:
Square:
Rectangle:
Triangle:
Trapezoid:
Circle:
Ellipse:
Hexagon:
Equilateral Triangle:

Volume:
Prisms (General):
Rectangular Prism:
Cylinder:
Cone:
Sphere:

Surface Area:
Triangular Prism:
Rectangular Prism:
Cylinder:
Cone:
Sphere:

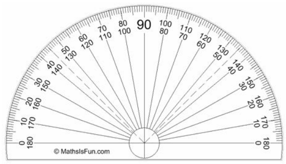
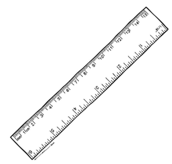
Circles:
Central Angles:
Intercepted Angles:
Chord-Chord Angles:
Secant-Secant Angles:
Secant-Tangent Angles:

Special Triangles:
30-60-90 Triangle:
45-45-90 Triangle:

Theorems:
Perpendicular Bisector Theorem:
Angle Bisector Theorem:
Pythagorean Theorem:

- Circles.
    - Central angles, inscribed angles, chord-chord angles, chord-tangent angles, secant-secant angles
    - Definition of circle, radius, chord, secant, tangent
    - Power of a Point
- Triangles.
    - Basic Congruence theorems (SSS, SAS, AAS/ASA, HL)
    - Basic Similarity theorems (AA, SSS, SAS)
    - Relationships of congruent/similar triangles with area
    - Angle Bisector Theorem
- Vocabulary.
    - Definition of perpendicular bisector, angle bisector
- Coordinates.
    - Definition of coordinates.
    - Equations of lines.
    - Intersections of lines.
    - Distance and Midpoint formulas.
    - Parallel and Perpendicular lines.
    - Proving points are collinear.

Geo 1.0
Basics (Reference Sheet)

To draw a precise diagram to solve a geometry problem, you use a **ruler**, **compass**, and **protractor.**

Fundamentals.

- A **ruler** draws circles, a **compass** creates straight lines, and a **protractor** creates accurate angles.
- A **line** connects two points. A **line segment** is the portion of a line between two points.
- An n-sided **polygon** is an ordered set of n points and the line segments between any two adjacent points (including the first and last one).
  - A polygon is **self-intersecting** if two of the line segments cross.
  - The **interior** of a non-self-intersecting polygon is the region bounded by the line segments.
  - We give special names to special polygons. From $n = 3$ to $n = 10$ we have the **triangle, quadrilateral, pentagon, hexagon, heptagon, octagon, nonagon, decagon.** n = 11 is the undecagon, and $n = 12$ is the **dodecagon.**
- The **length** of a line segment is the distance between its endpoints.
  - **Addition** of lengths: If A, B, C are on a line in that order, then $AB + BC = AC$.
- A **locus** of points is a set of points satisfying a given condition.
- A **circle** is the set of points at a fixed distance r from a fixed point, the **center.**
  - The set of points is also called the **circumference**, while the circle plus its interior is usually called a **disk.**
  - A **chord** (line segment) connects two distinct points on the circle.
  - A **secant** is the line formed by extending a chord outside the circle.
  - A **tangent** is a secant that intersects the circle at exactly one point. (Secants intersect the circle at two.)
  - An **arc** is the portion of the circumference between two points. There are two possible arcs between two points; the larger one is the **major arc**, and the smaller one is the **minor arc**.
- An **angle** is an ordered triplet of three distinct points.
  - The **vertex** is the middle point of the triplet.
  - The **measure** of an angle $\angle ABC$ is the amount needed to rotate line AB to get to AC.
  - The measure of a full rotation is $360°$. The measure of a half rotation, or a line, is $180°$.
  - Two angles are **complementary** if they sum to $90°$. Two angles are **supplementary** if they sum to $180°$.
  - You can add angles in the same way you add lengths.

    - An **interior angle** of a polygon is the angle inside the polygon. The **exterior angle** is the angle outside the polygon.
- Two distinct lines are **parallel** if they don't **intersect** at a point.
    - Euclid's 5th Axiom: Given a point A and a line l, there exists exactly one line through A parallel to l.
    - We use $l//m$ to say l is parallel to m.
- Two distinct lines are **perpendicular** if they intersect and the angle between them is $90°$. We say $l\perp m$ if l, m are perpendicular.

Basic Theorems. If you show one property, then the others follow immediately!

- If A, E, C are on a line in that order and D is an outside point, then $\angle AED$ and $\angle CED$ are supplementary.
- If A, E, C and B, E, D are on two lines in that order, then $\angle AEB$ and $\angle CED$ are **vertical angles**, so they are equal.
- **Parallel lines:** If $AB//CD$ and A, C are to the left of B, D:
    - $\angle ABC = \angle DCB$. (Alternate interior angles)
    - $\angle ABD + \angle CDB = 180°$. (Same side interior angles)
    - Any line perpendicular to AB is also perpendicular to CD.
- The **sum of the angles** in a triangle is equal to $180°$.
    - The sum of the angles of an n-gon is $180°(n-2)$.
    - The sum of the exterior angles of an n-gon is $360°$.
    - **Exterior angle theorem:** The sum of two interior angles of a triangle equals the exterior angle of the third vertex.
- The **perpendicular bisector** of AB, given distinct points A, B is the locus of points C such that:
    - If M is the midpoint of AB, then $CM\perp AB$.
    - ABC is isosceles with base AB. This implies $AC = BC$ and $\angle CAB = \angle CBA$.
    - CM is a symmetry line of triangle ACB.
    - A reflects to B upon reflection over CM.
    - Any point on line segment AC is closer to A than B, and any point on segment BC is closer to B than A.
- A **parallelogram** $ABCD$ is a quadrilateral with the following properties:
    - $AB//CD$, $AD//BC$ (opposite sides parallel)
    - $AB = CD$, $AD = BC$ (opposite sides equal)
    - If AC, BD meet at E, then $AE = CE, BE = DE$ (opposite sides bisect each other)
    - $\angle A = \angle C$, $\angle B = \angle D$ (opposite angles equal)
    - A translation sends AB to CD (which means $AB = CD$ and $AB//CD$)
- A **rectangle** $ABCD$ is a parallelogram with the following properties:
    - Adjacent sides are perpendicular. (or all angles are right angles)
    - Diagonals are equal.
    - $ABCD$ is cyclic.
    - If AC, BD meet at E, then $AE = BE = CE = DE$.

- A **kite** $ABCD$ with **symmetry diagonal** AC is a quadrilateral with the following properties:
  - $AB = AD, CB = CD.$
  - Diagonals are perpendicular ($AC \perp BD$)
  - AC is a line of symmetry for the kite.
  - AC is the perpendicular bisector of diagonal BD.
  - $\angle ABC = \angle ADC.$
  - AC **bisects** both $\angle BAD$ and $\angle BCD$ (so for instance $\angle BAC = \angle DAC$).
- A **rhombus** $ABCD$ is a parallelogram with the following properties.
  - All sides are equal.
  - It is a kite with both diagonals being symmetry diagonals.
- A **square** $ABCD$ is a parallelogram with the following properties.
  - It is a rectangle and a rhombus.
  - All sides are equal and all angles are right.
  - It is a regular quadrilateral.
  - If AC and BD intersect at E, then AEB, BEC, CED, DEA are congruent, isosceles right triangles.
  - Diagonals = √2 * side
- A **trapezoid** $ABCD$ with bases AB and CD is a quadrilateral with $AB//CD$.
- An **isosceles trapezoid** $ABCD$ is a trapezoid with the following properties:
  - $AC = BD$ (Diagonals are equal)
  - $AD = BC$ (**Legs** are equal)
  - $\angle ADC = \angle BCD$ (Base angles equal)
  - If AC and BD intersect at E, then $AE = BE, CE = DE$
  - $ABCD$ is cyclic
- A **cyclic quadrilateral** $ABCD$ is a quadrilateral with the following properties:
  - There exists a circle containing A, B, C, D on its circumference.
  - $\angle ABC = \angle ADC$ AND similar relations (inscribed arcs)
  - $\angle ABC + \angle ADC = 180^\circ$ (opposite angles are supplementary)
  - If AC, BD meet at E, then $AE \cdot CE = BE \cdot DE$ (Power of a Point, chord form)
  - If AB, CD meet at E, then $EA \cdot EB = EC \cdot ED$ (Power of a Point, secant form)
- Let AB be **tangent** to a circle centered at O with tangency point A. Then AB is perpendicular to OA.

Formulas for Special Shapes.

Type	Formula
Triangle	Sum of angles = 180° Area = $\frac{1}{2} \cdot bh$, b = base, h = height Area = $\frac{1}{2}ab\sin C$, a, b = adjacent sides to angle C Area = $\sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{a+b+c}{2}$ is **semi-perimeter**

	Area = rs, r = inradius Area = $\frac{abc}{4R}$, R = circumradius
Right Triangle $\angle C = 90°$ D = foot of altitude from C to AB M = midpoint of AB	$a^2 + b^2 = c^2$, c = hypotenuse (Pythagorean theorem) $\sin A = \frac{a}{c}$, a = opposite side to vertex A Area = $\frac{1}{2}ab$ Inradius = $\frac{a+b-c}{2}$ --------------- $AD \cdot c = b^2, BD \cdot c = a^2$ $AD \cdot BD = CD^2$ --------------- $AM = BM = CM$ M lies on perpendicular bisectors of sides --------------- Special Right Triangles 45-45-90: sides in ratio 1-1-$\sqrt{2}$ 30-60-90: sides in ratio 1-$\sqrt{3}$-2
Quadrilateral	Sum of angles = $360°$ Area of cyclic quad = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, $s = \frac{a+b+c+d}{2}$ is semi-perimeter
Polygon	Sum of angles = $180°(n-2)$ Find area by: (i) splitting into many smaller figures (e.g. triangles) which we know area of (ii) taking the complement and finding area of that
Circle	Area = πr^2 Circumference = $2\pi r$ Area of sector = $\frac{1}{2}\theta r^2$, θ in **radians** Circumference of sector = θr, θ in radians (2π radians = 1 circle = $360°$)

Tips

- **Exploit symmetry.** Some properties I included relate to symmetry. Always try to look for symmetry, because you can "intuitively" guess corresponding things are equal, which can lead to a solution for the problem.
- **Look for similar/congruent triangles.** Almost all of the properties can be deduced from the correct similar/congruency relations.
- **Drop perpendiculars.** Especially when you have tangency points, dropping perpendiculars can create right triangles which are good for lots of things, such as similar triangles and Pythagorean theorem.

Notes 1.1 Name: ____________________
Analysis

There are three main techniques to progress on a problem: length bash, angle chasing, and synthetic observations. We will be focusing on length bash and angle chasing here.

Length bash:

- Addition of lengths
  - If A, B, C are on a line on that order, then AB + BC = AC
- Pythagorean theorem
  - Drop altitudes to create right triangles
  - This is especially useful in isosceles triangles because the foot of the altitude is also the midpoint of the side

Angle chasing:

- Look for 30-60-90 and 45-45-90 right triangles
- Look for equal angles in isosceles triangles
- Sum of angles in a triangle
- Parallel lines

General tips:

- Set variables to unknown lengths and solve for them using equations
- Connect tangency points to centers in circles (this creates right triangles)
- Connect centers in tangent circles; this line passes through the tangency points
- Areas: Some problems ask to find areas of difficult regions. In that case, try adding or subtracting easy-to-compute areas to get the desired area. There might be more than one way to do so, and some ways are easier than others.

Use your theorems! Geometry is basically making the right insight at the right time.

Basic exercises.

1. Draw a parallelogram, isosceles trapezoid, rectangle, square, and rhombus that is not a square. Draw in diagonals and label all equal sides and angles.
2. Three circles of radii 2 are mutually tangent. What is the area enclosed by the circles?
3. Four lampposts A, B, C, D are on a road in that order. If B is equidistant from A and C, C is equidistant from A and D, and BD = 6, find AC.
4. In quadrilateral ABCD, we have <ABC = <ADC = 90°, <DAB = 120°, AB = 1, and CD = 3. Find <BCD and the area of ABCD. (Hint: look for right triangles.)
5. What is the sum of the vertex angles in a 5-pointed star?
6. A triangle ABC has side lengths 5, 12, 13 and hypotenuse AC. If I is the incenter of ABC, what is BI?
7. Two circles of radii 8 and 13 are externally tangent. Find the length of the external tangent.

8. In right triangle ABC, the midpoint of AB is M, the midpoint of BC is N, and the midpoint of hypotenuse AC is K. If AN = 7 and CM = 8, find the length of BK.
9. In triangle ABC, the angle bisectors of <ABC and <ACB meet at I. If <BAC = 70°, find <BIC.
10. Two circles of radii 8 and r > 8 are internally tangent at P. The larger circle is centered at O, and tangent AB (with A and B on the larger circle) is drawn to the smaller circle such that AB is perpendicular to OP. If AB = 24, find r.
11. Given segment AB, explain why the set of all points C such that [ABC] = k (where k is a fixed constant) is two lines parallel to AB.

Parodied 2016 Mathcounts problems.

1. An isosceles triangle ABC has vertex angle <A = 44°. Point P is located on the same side of B of AC and PA // BC, <APB = 32°. Find <PBA.
2. In the same isosceles ABC, BE is a bisector of <ABC. Find <AEB.
3. Triangles ABC and ACD are isosceles right triangles sharing AC. What is the greatest possible ratio of areas [ABC]/[ACD]?
4. Given triangle ABC, point D is on BC such that <ADB = 60°. The incircle of ABD intersects AD at E, and the incircle of ACD intersects AD at F. If AE = 4, EF = 2, FD = 1, find the ratio of areas of the two circles.
5. A room is partitioned into 5 smaller congruent rooms by a line drawn straight down the middle of the large room and three horizontal lines that end at the vertical line. What is the ratio of side lengths of the big room?

Weird Areas Practice.

1. Point D is on side BC of triangle ABC. Explain why [ABD]/[BCD] = BD / CD. (This is known as same base, same height.)
2. Points E, F are on sides AC, AB. Justify each step:

$$[AEF]/[ABC] = [AEF]/[AEB] \cdot [AEB]/[ABC] = AF/AB \cdot AE/AC.$$

3. On sides BC, CA, AB of triangle ABC locate points D, E, F respectively such that BD = 2DC, CE = 2EA, AF = 2FB. Determine the ratio of areas [DEF] / [ABC].
4. Given a cyclic quadrilateral with sides subtending arcs of measure 60°, 90°, 90°, 120° in that order, find its area in terms of these measures and the radius R of its circumcircle.
5. Find the area of a sector with measure θ, where θ is in degrees.
6. Find the circumference of a sector with measure θ, where θ is in degrees.
7. Triangle ABC with side length 2 is equilateral. A circle is centered at C and passes through A. What is the area of the region bounded by AB and the circle's circumference?
8. Use the configuration from problem 7. A semicircle has diameter AB and intersects the exterior of the circle. What is the area of the region inside the semicircle and outside the circle?
9. Two unit circles are centered at $(0, 0)$ and $(0, 1)$. Find the area of their intersection.

Lesson 1.2 Name: _______________
Triangles

A triangle has 3 vertices and 3 sides. Usually we work with triangle ABC. We also work with two triangles; we aim to prove they are similar or congruent. Draw and label corresponding parts of triangles ABC and DEF.

Congruency: ABC ___ DEF if and only if:

Similar Triangles: ABC ___ DEF if and only if:

<u>**Congruency**</u>
SSS Congruence:

SAS Congruence:

AAS/ASA Congruence:

SSA Congruence:

<u>**Similarity**</u>
AA Similarity:

SAS Similarity:

SSS Similarity:

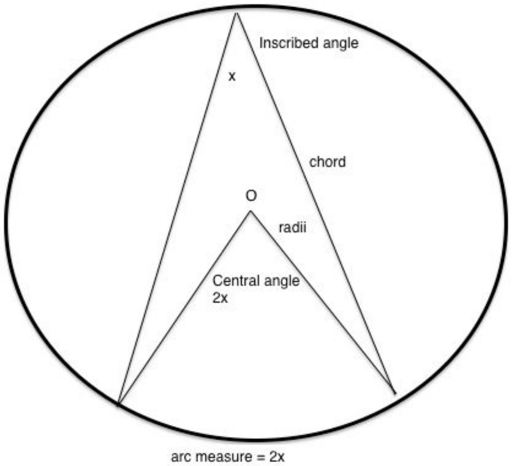
Exercises.

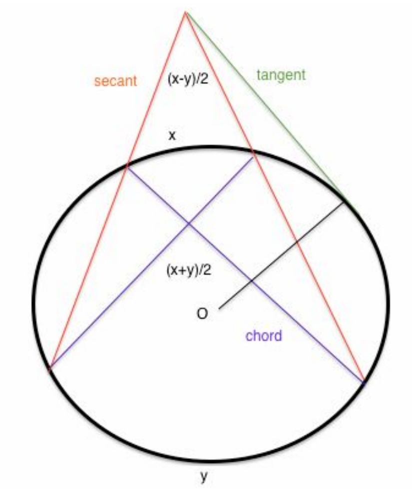
1. Given rectangle ABCD, list all pairs of distinct, congruent triangles whose vertices are in A, B, C, D. (Hint: There are 6.)
2. Triangles ABC and ADE are such that BC // DE. What two triangles are similar? If AB = 10, BD = 8, and AC = 7, what is CE?
3. Triangle ABC is given. Prove that AB = AC if and only if <B = <C. (Hint: Drop an altitude and use congruence. Remember you have to prove two implications!)
4. In triangle ABC, construct **medians** AD, BE, CF, where D, E, F are the midpoints of BC, CA, AB, respectively. It turns out they all meet at a point G, called the **centroid.**

   a. Prove that EF = BC / 2.
   b. Prove that AG / GD = BG / GE = CG / GF = 2.
5. Define the perpendicular bisector of segment AB to be the set of points C such that $CM \perp AB,$ where M is the midpoint of AB. Prove that C is on the perpendicular bisector of AB if and only if AC = BC.
6. Triangles ABC and ADE are such that <ABC = <AED. If AB = 10, AC = 20, AD = 15, and DE = 20.14, what is AE?
7. Segments AB and CD intersect at E such that AE * BE = CE * DE. List all pairs of congruent angles.
8. ABCD is a trapezoid with AB // CD. Diagonals AC and BD intersect at E. If AB = 5, CD = 10, and AC = 18, what is AE?
9. Triangles ABC and DEF are congruent. Are their areas the same? Explain.
10. Triangles ABC and DEF are similar such that AB / DE = k. Find [ABC] / [DEF], where [R] is the area of region R.
11. *(Sycamore Ridge Qualification Test)* Trapezoid ABCD has a top base that is 5/7 of its bottom base. The diagonals AC and BD of the trapezoid are drawn and meet at point E. What is [ABE] / [CDE]? Express your answer as a simplified fraction.
12. A quadrilateral ABCD is a parallelogram if AB // CD and BC // AD. Prove that ABCD is a parallelogram if and only if AB = CD and BC = AD. Also, if the diagonals AC and BD meet at E, prove that ABCD is a parallelogram if and only if AE = CE and BE = DE.
13. In triangle ABC, D is on BC such that <BAD = <CAD. (AD is an angle bisector.)
    a. If AB = AC, prove that BD = BC.
    b. Point E is on AC and F is on AB such that $DE \perp AC$ and $DF \perp AB$. Prove that $DE = DF.$ Conversely, if $DE = DF$, then show that $\angle BAD = \angle CAD.$
    c. Similarly define E on AC such that BE is an angle bisector. Let AD and BE intersect at I. Prove that if X, Y, Z are the projections from I onto the sides, then IX = IY = IZ.
    d. Define P on ray AD such that AB // CP. Prove that AB / BD = PC / CD. Deduce the Angle Bisector Theorem: $\frac{AB}{AC} = \frac{DB}{DC}.$
14. Classify the following quadrilaterals ABCD (i.e. list the most specific type of quadrilateral ABCD must be):
    a. AB // CD, AB = CD
    b. AB // CD, AC = BD
    c. AB // CD, AD = BC
    d. AB = AD, AC perpendicular to BD
    e. AB = AD, AC bisects <BAD
    f. AC is bisected by BD, AC perpendicular to BD
15. In parallelogram ABCD, M is the midpoint of side CD. Lines AM and BD intersect at P. Compute the ratio AP / PM. (Hint: extend AM to meet line BC at X. Then use similar triangles.)
16. In trapezoid ABCD with AB // CD, we have AB = 7, CD = 21, BD = 28. Compute BP, where P is the intersection of diagonals AC and BD.

Lesson 1.3 Name: ___________________

Circles and Weird Areas





A **central angle** is bounded by radii. An **inscribed angle** is bounded by chords that meet on the circle.

Theorem. The arc measure is defined to be the measure of the central angle corresponding to that arc. Then the inscribed angle is half the arc measure.

Theorem 2. An angle intercepted by two chords that intercept arcs with measures x and y has measure (x + y) / 2.

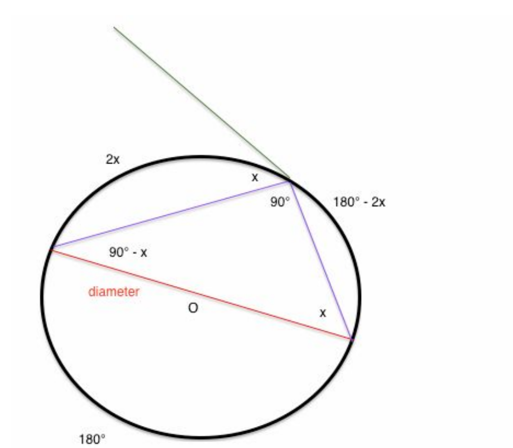
Theorem 3. An angle intercepted by two secants that intercept arcs with measures x and y (with y > x) has measure (y - x) / 2.

Theorem 4. The same theorem holds for secant-tangents.

A full circle has arc measure 360°. A semicircle thus has arc measure 180°. By the inscribed angle theorem, any angle inscribed in a semicircle is equal to 90°.

Theorem 5. A chord and a tangent intersect at an angle equal to half the intercepted arc's measure.

The **circumcenter** of a triangle ABC is the point O such that OA = OB = OC. The circle with center O and radius OA is the **circumcircle** of ABC. We can use circle properties with the circumcircle.



Lengths
Use similar triangles to prove:

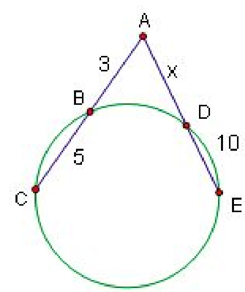
Exercise 6. If AB and CD are chords of a circle meeting at P, then AP * BP = CP * DP.

Exercise 7. If AB and CD are secants of a circle meeting at P, then PA * PB = PC * PD.

Exercise 8. If PA is a tangent and PBC is a secant, then PA^2 = PB * PC.

Problem 1

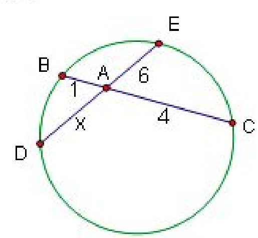
Find the value of x in the following diagram:



Solution

Problem 2

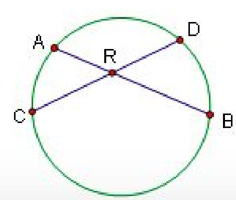
Find the value of x in the following diagram:



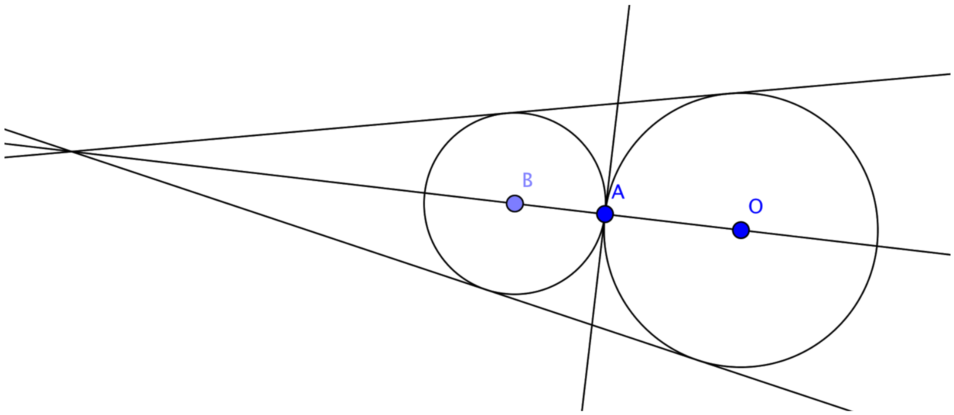
Solution

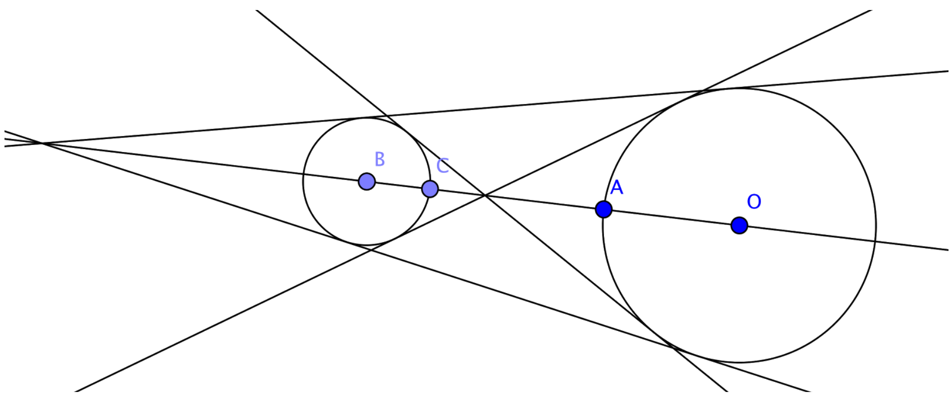
Problem 3

(ARML) In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.



Tangent Circles and Common Tangents





Problems.

1. Prove that any angle inscribed in a semicircle is a right angle.
2. Triangle ABC has circumcenter O. If <BAC = 60°, find <OBC.
3. Define point E on AC such that $\angle CDB = \angle ADE$. Show that $AB \cdot CD + AD \cdot BC = (AE + EC) \cdot BD = AC \cdot BD$.
4. $ABCDEFGHIJKL$ is a regular 12-gon. Find $\angle FAD$ and $\angle LEG$.
5. In a triangle ABC with circumcircle ω, consider the intersection of angle bisector AI (I is the incenter) with ω, and call it M. Show that M is the circumcenter of BIC.
6. Let D is the foot of the angle bisector.
   a. Show that $ABD \sim AMC$ by AA Similarity.
   b. Find another triangle ABD is similar to.
7. In rectangle ABCD, M is the midpoint of AB. Point X is such that MX = MC. If <XMC = 83°, find <XDC. (Source: AoPS)
8. In cyclic quadrilateral ABCD, diagonals AC and BD are perpendicular, and AB and CD intersect at E. If <AEC = 30°, find <ABD.

Tangent circles

9. Given two externally tangent circles of radii R and r, find the length of their external common tangent.

10. A 12-gon has a circle inscribed inside of it and circumscribed about it. Find the area between the two circles in terms of the side length *s* of the 12-gon.
11. Given two externally tangent circles ω_1 and ω_2 that are tangent at B, consider $A \neq B$ on ω_1 and $C \neq B$ on ω_1 such that A, B, C are collinear. Prove that $\widehat{AB}$ in ω_1 has the same measure as $\widehat{BC}$ in ω_2.
12. Point P is outside circle O, and PA and PB are tangents to circle O (A and B are on the circle). Prove that $PAOB$ is a cyclic kite. In particular, we have PA = PB (equal tangents).
13. Given two externally tangent circles centered at A and B that are tangent at P, let CD be a common external tangent with C on circle A and D on circle B. Show that $\angle CPD = 90^\circ$. (Hint: there are other things true in the diagram.)

Notes 1.4 Name: ____________________

Coordinate Geometry

Draw a coordinate system below.

<u>Points and Lines</u>

x-intercept, y-intercept:

Equations of Horizontal Lines:

Vertical Lines:

Slope:

Slope-Intercept Form:

Point-Slope Form:

Standard Form:

<u>Concerning Two Points</u>

Midpoint Formula:

Distance Formula:

<u>Concerning Two Lines</u>

Parallel Lines have ___________ slope.

Perpendicular Lines have slopes that multiply to _____.

To find the intersection of two lines, you solve the system of equations of these lines.

Exercises.

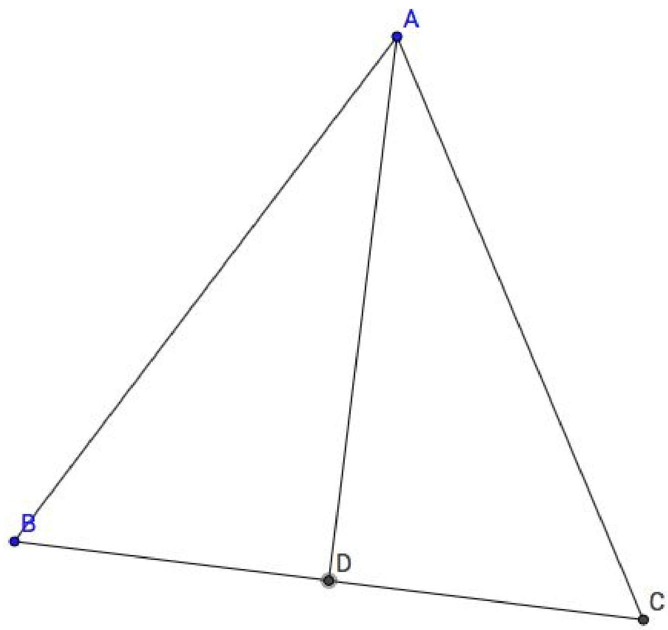
1. Find the equation of the line passing through (2, 5) and (12, 7) in standard form.
    a. What is the slope of the line? ____
    b. What is the equation of the line in point-slope form? ___________________
    c. What is the equation of the line in standard form? _____________________
2. Find the equation of the line passing through (1, 7) and (2, 5) (in any form).
3. Find the equation of the line passing through (0, 5) and (7, 2014).
4. Find the intersection of lines $y = 2x + 5$ and $y = 3x - 4$.
5. Find the intersection of lines $y = 3x - 7$ and $2x + 5y = 17$.
6. Given A(1, 4), B(2, 9), C(6, 8), D(4, 3): *(8 points)*
    a. Compute AB, BC, CD, DA.
    b. Find the intersection of diagonals AC and BD.
    c. Find the midpoint of:
        i. AB.
        ii. BC.
        iii. AC.
        iv. The segment formed by the midpoints of AB and CD.
    d. Determine the equation of the line formed by the intersection of sides AB and CD and by the intersection of sides AD and BC.
7. Given quadrilateral ABCD, the midpoint of AB is E, of BC is F, of CD is G, of DA is H, of AC is I, and of BD is J. Prove that the midpoints of EG, FH, and IJ are the same point.
8. The **perpendicular bisector** of segment AB is perpendicular to AB and passes through the midpoint of AB. *(8 points)*
    a. Find the perpendicular bisector of (1, 24) and (7, 12).
    b. Find the perpendicular bisector of (0, 0) and (13, 27).
    c. Find the perpendicular bisector of (x, y) and (z, w).
9. Determine the equation of the line through (2, 4) parallel to $3x + y = 17$.
10. Determine the equation of the line through (2, 4) perpendicular to the line through (10, 24) and (16, 30).
11. Given A(0, 0) and B(12, 36), find both locations for C on line AB such that AC = 2CB. Verify this using the distance formula.
12. Given A(5, 17) and B(20, 26), find both locations for C on line AB such that 2AC = CB.

Theorem. Points A, B, C are collinear if and only if slope(AB) = slope(AC) = slope(BC). You only have to prove one equality for the points to be collinear.

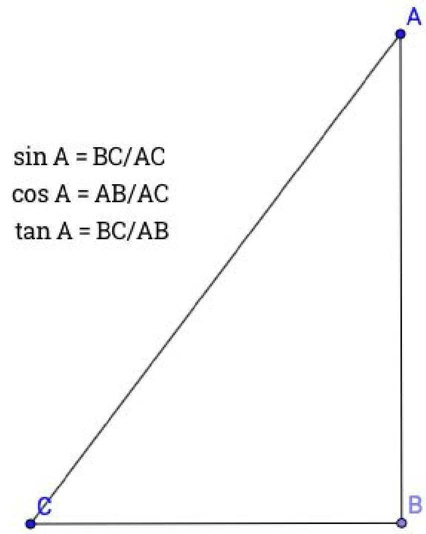
13. Determine if the following points are collinear:
    a. A(2,3), B(3,5), C(4,7)
    b. A(1,7), B(2,6), C(3,4)
    c. A(2014,2014), B(1006,1006), C(5,5)
    d. A(12,4), B(3,7), C(0,8)
    e. A(4,6), B(6,8), C(12,14)
    f. A(5,2), B(1,6), C(-3, 10)

Notes 1.5 Name: ____________________
Trigonometry

<u>Directions</u>. Solve the following problems. Point values for each problem are listed after each problem. Show work for all problems to receive full credit.



1. Triangle ABC is equilateral with side length 1.
    a. Altitude AD is drawn (such that D is on BC). What two triangles are congruent, and by what theorem? List vertices in corresponding order. [3]
    b. Find AD, BD, and CD. [3]
    c. Find <DAC and <C. [2]
2. Triangle ABC is isosceles with AB = BC = 1 and <B = 90°.
    a. Find AC. [2]
    b. Find <A and <C. [2]
3. Triangles ABC and DEF are right-angled at <B and <E, respectively, such that <A = <D.
    a. What two triangles are similar? List vertices in corresponding order. [2]
    b. Show that BC/AC = EF/DF. [2]
    c. Define the *sine* of an angle, sin <A, to be the opposite side (to <A) divided by the hypotenuse in right triangle ABC. Prove that the sine of a given angle is constant, no matter what triangle <A is part of. (Hint: Use part (b).) [4]

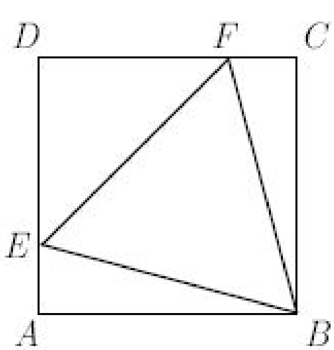


d. cos and tan functions are defined similarly, as in the figure. Show that cos <A and tan <A also are only dependent on <A. [4]
e. Complete the following table. You do not need to show work. [9]

	sin	cos	tan
30°			
45°			
60°			

f. Prove that $tan\ A = \frac{sin\ A}{cos\ A}$ and $sin^2 A + cos^2 A = 1$ for all acute angles A. [8]
g. Prove that $sin\ A = cos\ (90° - A)$, $cos\ A = sin\ (90° - A)$, and $tan\ A\ tan\ (90° - A) = 1$ for all acute angles A. (Hint: what is 90° - A in a right triangle ABC?) [9]

4. (AMC 10) Points E and F are located on square *ABCD* so that *BEF* is equilateral. What is the ratio of the area of *DEF* to that of *ABE*? [10]



In this Mystery Problem Set, you deduced the ratios of the sides of 30-60-90 and 45-45-90 triangles. You also proved that such ratios are constant and do not vary from triangle to triangle.

5. In triangle ABC, AB = $\sqrt{3}$, BC = 1, and CA = 2. Point D is on BC such that <BAD = <CAD. Find (a) BD; (b) tan 15°; (c) sin 15°.

PTMS MATH LEAGUE

CHAPTER TWO

ALGEBRA

Notes 2.0 Name: ______________________

Basics

To solve a linear equation, you isolate the variable.

3x + 5 = 4x + 9 -> -4 = x

4(x + 2) = 7(x + 5) -> 4x + 8 = 7x + 35 -> -27 = 3x -> x = -9

FOILing (First, Outside, Inside, Last):

(x + 3)(2x + 1) = 2x^2 + x + 6x + 3 = 2x^2 + 7x + 3.

(2x - 1)(x + 4) = 2x^2 + 8x - x - 4 = 2x^2 + 7x - 4.

Factoring: Use the X-Factor or a similar method.

x^2 + 6x + 5 = (x + 5)(x + 1).

2x^2 + 7x + 6 = (2x + 3)(x + 2).

Remember to factor out common factors while factoring.

3x^2 + 3x - 6 = 3(x^2 + x - 2) = 3(x + 2)(x - 1).

2014x^2 + 4028x + 2014 = 2014(x^2 + 2x + 1) = 2014(x + 1)^2.

Exercises. Complete the following problems.

Factored Expression *(Simplify the following expressions.)*

1. (2x + 5)(3x - 5) =
2. (x + 4)(x - 3) =
3. (2x - 6)(3x - 7) =
4. (x - 4)(3x + 3) =
5. (3x + 1)(x + 7) =
6. (x - 7)(3x - 1) =
7. (x + 4)(x + 5) =
8. (3x + 6)(2x - 2) =
9. (3x + 6)(x + 6) =
10. (2x - 1)(3x - 6) =
11. (3x + 7)(3x + 1) =
12. (3x - 3)(x + 6) =
13. (x - 3)(3x - 5) =
14. (2x - 7)(x - 2) =
15. (2x + 2)(3x - 5) =
16. (2x + 2)(3x + 5) =
17. (x + 2)(2x + 7) =
18. (2x - 2)(3x + 2) =
19. (3x - 5)(3x + 3) =
20. (2x + 1)(2x + 4) =
21. (3x + 5)(x + 5) =
22. (3x - 3)(2x + 7) =

23. $(3x + 1)(3x - 3) =$
24. $(2x + 4)(2x + 5) =$
25. $(2x - 4)(3x - 7) =$
26. $(3x - 6)(3x + 1) =$
27. $(2x - 7)(3x - 3) =$
28. $(3x + 5)(2x + 7) =$
29. $(3x + 7)(2x - 3) =$
30. $(x + 1)(2x - 2) =$

FOILed Expression *(Factor the following expressions.)*
1. 4x^2 + 10x + 6 =
2. 4x^2 - 24x + 36 =
3. 6x^2 - 5x + 1 =
4. 6x^2 - 7x - 49 =
5. x^2 - 11x + 30 =
6. 2x^2 + 11x + 14 =
7. 3x^2 - 18x + 24 =
8. 3x^2 + 21x + 30 =
9. 2x^2 - 8x - 10 =
10. 6x^2 + 22x + 20 =
11. 9x^2 - 9x - 18 =
12. 9x^2 + 27x + 20 =
13. 9x^2 + 15x - 6 =
14. x^2 - 8x + 12 =
15. 3x^2 + 13x - 10 =
16. 3x^2 + 11x + 6 =
17. 2x^2 - 3x - 35 =
18. 2x^2 - 5x - 42 =
19. 2x^2 - 3x - 20 =
20. 3x^2 - 5x - 2 =
21. 4x^2 - 4 =
22. 4x^2 - 4x - 15 =
23. 2x^2 - 15x + 7 =
24. 9x^2 + 3x - 42 =
25. 9x^2 + 9x - 28 =
26. 6x^2 - 9x - 6 =
27. 4x^2 - 6x - 4 =
28. 6x^2 - 25x + 25 =
29. x^2 + 2x - 8 =
30. 6x^2 - 6x - 12 =

Factoring is useful for solving equations. If you have ac = ad, then either a = 0 or c = d. This is equivalent to “**canceling**” the *a*.

Notes 2.1 Name: ______________________

d = rt and Work Problems

Distance = rate * time.

Example 1. I drive a distance at 60 miles per hour in 40 minutes. How many minutes will it take David to drive the same distance at 40 miles per hour?
Solution. The distance equals (60 miles/1 hour * 1 hour/60 minutes) * 40 minutes = 40 miles. Thus, it will take David (40 miles) / (40 miles/hour) = 1 hour = 60 minutes to drive the same distance.

Always set equations in d = rt problems!
Example 2. I usually walk to school in 15 minutes. However, on one day, I had to walk to school in 10 minutes to arrive on time. Given that my rate increased by 3 m/s, what is the distance between my house and school?

Solution. Let r = original rate, d = distance.

Then d = r * 15 * 60 = (r + 3) * 10 * 60, so 3r = 2(r + 3) and r = 6. Then d = 5400 m.

Exercise 1. The tortoise and the hare are having a rematch. The tortoise gets a 50 m head start, while the hare decides the length of the race. The tortoise walks at 3 m/s, while the hare runs at 5 m/s. What is the minimum integral length such that the hare will win (and not tie)?

Average Speed = total distance / total time.

Example 3. I walk up a hill at 3 m/s and down the hill at 6 m/s. What is my average speed?
Solution. Let the length of the hill be d. Then my total time is d/3 + d/6 = d/2. My total distance is 2d. Hence, my average speed is 4 m/s.

Exercise 2. David walks halfway to school at 4 m/s when he realizes he forgot his homework! He runs back to his house and then to school at 12 m/s. What is David’s average speed?

Exercise 3. In a 60-mile triathlon, Roger bikes at 15 mph for 2 hours, swims at 3 mph in the 10-mile swim, and finally runs at 12 mph. What is Roger’s average speed in the triathlon?

Remember that when two forces are acting on an object at the same time, the rates ______.
Example 4. A man and his dog plow a field. The man can plow a field in 3 hours, while a dog can plow a field in 2 hours. If they work together, in how many hours can the man and dog plow the field?
Solution. The rate of the man is 1/3 fields/hour. The rate of the dog is 1/2 fields/hour. Thus, together they plow 5/6 fields/hour. To plow one field, they need 6/5 hours.

Exercise 4. A man drives to work at 40 miles per hour and arrives 15 minutes late each day. After his boss got mad at his tardiness, the man increased his speed to 60 miles per hour and now arrives 15 minutes early each day. What should the man's speed be to arrive exactly on time?

Exercise 5. Three men decide to read *Pride and Prejudice* together. If the first man can do so in 1 hour, the second man can do so in 2 hours, and the third man can do so in 3 hours, how many hours will it take them to read the *Merriam-Webster Dictionary,* which takes ten times as long to read?

Exercise 6. A bathtub is currently empty. Two faucets A and B are simultaneously turned on. Faucet A can fill a tub in 40 minutes, and Faucet B can fill a tub in 60 minutes. However, the drain can empty a tub in 45 minutes. In how many hours will the tub be filled?

Work Problems.
Work = rate * time * quantity.

Example 5. Two moles dig two holes in two hours. How many moles are required to dig three holes in three hours?
Solution 1. We create a Work chart:

Rate	Quantity	Time	Work
--	2	2	2
--	x	3	3

We have x/2 * 3/2 = 3/2, so x = 2.

Solution 2. Because W = rtq, we have 2 = 2 * 2 * r, so r = 1/2. Hence, if 3 = 3 * (1/2) * q, then q = 2.

Example 6. Three atoms generate 30 W of energy in thirty minutes. How many quarks would it take to generate 210 W of energy in an hour, given that a quark produces half as much energy as an atom?

Solution. Let r = rate of atom, R = rate of quark. Then W = rtq -> 30 = r * 1/2 * 3, so r = 20. Hence R = 10. Then W = Rtq results in 210 = 10 * 1 * q, so q = 21.

Exercise 7. If 10 woodchucks can chuck 24 pounds in 48 minutes, how many woodchucks are required to chuck 240 pounds in 24 minutes?

Exercise 8. Ten 6th graders can solve 35 problems in one hour. If 7th graders work twice as fast as 6th graders, 8th graders work three times as fast as 7th graders, and Kevin works four times as fast as an 8th grader, how many more problems can 40 6th graders, 30 7th graders, and 20 8th graders solve in three hours than one Kevin in 24 hours? (Your answer could be negative.)

Even clock problems are just distance = rate * time.

Example 9. What is the rate of the minute and hour hands? (i.e. after one minute, how many degrees have each hand moved?)
Solution. The minute hand travels 360° in 60 minutes, so 6°/min. The hour hand travels 1/12th of a clock, or 30° in 60 minutes, so 0.5°/min.

Exercise 10. If the minute hand moves (a) 90°; (b) 42 minutes, how many degrees has the hour hand moved?

Exercise 11. Find the angle between the minute and hour hand at (a) 3:00; (b) 3:15; (c) 4:23; (d) 8:20; (e) 11:05. (Remember if the angle is greater than 180° you have just measured the reflex angle; subtract that from 360° to get an answer.)

Percent problems are a rather simple application of distance = rate * time.

Exercise 12. A bottle of beer contains 30% alcohol. How much alcohol do 99 bottles of 1 quart beer contain?

Exercise 13. A solution contains 50 mL of water per 1 L of solution. What percentage of the solution is not water?

When mixing solutions, the **quantities**, not the rates, add. (Mostly this is common sense: 1 mL of water + 1 mL of water = 2 mL of water. On the other hand, when two faucets run at the same time, then the rates add.)

Example 14. How much water do I need to add to 10 L of 30% acid to make the solution 20% acid?
Solution. There is 10 * 0.3 = 3 L acid initially. We add water, so acid doesn't change. We want 20% acid, so 3 / 0.2 = 15 L solution total. Thus add 15 - 10 = 5 L of water.
Alternate Solution. Let x L of water be added. Then 3 L of acid, so 3 / (10 + x) = 1 / 5. Thus 10 + x = 15, so x = 5.

Exercise 15. What percent acid occurs when we mix 10 L of 20% acid with 20 L of 50% acid?

Mean = sum of elements / total number of elements. Median = middle element or average of two middle elements. Unique mode = number that occurs most often. Range = largest element - smallest element.

Example 16. Find the mean, median, mode, range of 1, 1, 2, 3, 5, 8.
Solution. Mean = 20 / 6 = 10/3.
Median: We cross off biggest and smallest, left with 1, 2, 3, 5. Repeat, left with 2, 3. Median = 2.5.
Mode: 1 by a long shot
Range: 8 - 1 = 7

Lots of problems will ask you to find an element of a set given restrictions on mean, median, mode, or range. They are annoying because you have to deal with restrictions one at a time.

Example 17. The mean of Sarah's 6 test grades is 88. What is the least grade she needs to bring her average grade up to a 89.5?
Solution 1. Her sum of test grades is 88 * 6 = 528. In 7 tests she needs 89.5 * 7 = 626.5. Thus she needs 626.5 - 528 = 98.5.
Solution 2. She lacks 1.5 on average for 6 tests, so she lacks 9 total. She needs 89.5 + 9 = 98.5 to compensate.

Example 18. The mean of 6 distinct positive integers is 16. What is the largest possible value for the largest of them?
Solution. Their sum is 96. The smallest 5 should be 1, 2, 3, 4, 5, so the largest is 81.

Exercise 19. The mean of 6 distinct positive integers is 15 and their median is 10. What is the largest and smallest possible value for the largest of them?

Exercise 20. The mean of 100 numbers is 2016. The mean of 200 other numbers is 2019. What is the mean of all 300 numbers?

Exercise 21. A list has 8 integers with mean 4k. Adding one integer raises the mean by k. What is the ratio of the added integer to the sum of the 8 integers?

Notes 2.2 Name: ____________________

Special Algebraic Manipulations, Part 1

Solve for all complex (real and non-real) solutions unless otherwise specified. Leave your answers in exact form.

Adding Equations

Sometimes adding and subtracting equations can solve problems.

1. If a + b = 10 and a - b = 4, find a + 2b.
2. If a + b = 80, b + c = 40, c + a = 20, find a + b + c.
3. If a + 3b = 50, a + b = 70, find a + 2b.
4. If a + 3b = 100, a + 2b = 120, find a + b.
5. Ten apples and a pear cost $80. Ten pears and an apple cost $74. Find the cost of 2016 apples and 2016 pears.

Cancellation

If ab = 0, then a = 0 or b = 0. In other words, we can cancel a from the equation. This is why we factor equations.

1. Solve $(x^2-2x-3)(x^2-5x-6)=0.$
2. Solve $(x-8)(x^2-5x-9)=x^2-7x-8.$
3. Solve $2x^{2001}=5x^{2000}(x+8).$
4. Solve $(x^2-3x+2)(x^2+18x-9)=(x^2+4x-5)(x^2+8x-20).$
5. Solve $x^3-125=2(x^4-22x^2-75).$

Taking an Equation to a Power

If $a=b$, then $a^2=b^2$, $a^3=b^3$, $a^n=b^n$, $\sqrt{a}=\sqrt{b}$ (a, b > 0), etc. **Remember that** $\sqrt{a^2}=|a|$ **!!!** Also, $a^2=b^2$ implies $a=\pm b$, but $a^3=b^3$ implies $a=b$.

Example. If $x+\frac{1}{x}=4$, find $x^2+\frac{1}{x^2}$.

Solution: Squaring yields $x^2+2+\frac{1}{x^2}=16,$ so the answer is 14.

1. If $x+x^{-1}=5$, find x^4+x^{-4}.
2. Compute $\sqrt{(-2014)^2}+\sqrt{2014^2}$.
3. If $x^2+\frac{1}{x^2}=14$, find all possible values of $x-x^{-1}$.
4. If $x^4+x^{-4}=47$, find all possible values of x^1-x^{-1}.
5. Solve for x: $x+3=\sqrt{x^2+12x}$.
6. Solve for x: $3x-5=\sqrt[3]{27x^3+8x+1}$.
7. (a) Solve for x: $\sqrt{3x+1}+4x=24$. **Be careful of extraneous solutions!**
   (b) Solve for x: $\sqrt{5x+3}+7=\sqrt{5x+9}$.

Example 2. Solve for x: $x^2=(x+1)^2$.

Solution: Taking the square root of both sides yields $x = x + 1$ or $x = -1 - x$. The first equation has no solutions, while the second has solution x = -1/2, which is our answer.

8. If $(x+5)^{100} = (x-6)^{100}$, find x.
9. If $(2x^2 + 6x - 5)^{2015} = (x^2 - 5x - 7)^{2015}$, find x.
10. If $x^2 + x^{-2} = 14$, find both values of $x^3 + x^{-3}$.
11. If $x^{24} + 1 = 194x^{12}$, find all possible values of $\frac{x^6-1}{x^3}$.
12. If a number equals six plus its square root, what are all possibilities for that number?
13. (a) Expand and simplify $(x+1)^3$.
    (b) Solve for real x: $-26x^3 + 3x^2 + 3x + 1 = 0$. *(Source: AIME)* (Hint: How does part (a) help?

Manipulating Exponents

If $2^x = 2^y$, then $x = y$. Also, remember the exponent rules:
$(a^b)^c = a^{bc},\ a^b * a^c = a^{b+c},\ a^{b-c} = \frac{a^b}{a^c}.$

Example. If $2^{3x} = 2^4$, find x.

Solution. We have 3x = 4, so x = 4/3.

1. If $2^{2^x} = 2^{2^y}$, prove that x = y.
2. If $2^x = 1/4^8$, find x.
3. If $2^{3x+5} = 8^{-23} * 4^x$, find x.
4. If $3^{9x} = 9^{24}/81^{14}$, find x.
5. If $(4^x)^x = 2^{x-5} * 8^{3x+7}$, find x.
6. If $125^x = 25^{x^2-5x}/5^{30x}$, find x.
7. A bacteria colony doubles its population every 10 seconds. After x seconds, the population increases from 1 bacterium to 2^{20} bacteria. What is x?

Some Useful Substitutions

Example. Solve for x: $x^4 + 2x^2 + 1 = 0$.

Solution. Let $u = x^2$. Then $u^2 + 2u + 1 = 0$ becomes $(u+1)^2 = 0$, so $u = -1$ and $x = \pm i$.

Alternate Solution. We have $(x^2+1)^2 = 0$, so $x = \pm i$. Many equations are in such *quadratic form* and thus can be factored like a quadratic.

Example 2. Solve for x: $(x + 2015)^2 = 2x + 4029.$

Solution. One may expand and solve this quadratic directly, but there is a much faster solution. Let $u = x + 2015;$ then $u^2 = 2u - 1.$ This becomes $u^2 - 2u + 1 = 0$ and now one recognizes the factorization $(u-1)^2 = 0$, so $u = 1$ and $x = -2014.$

1. Solve for all real x: $x^{4028} - 5x^{2014} + 6 = 0$.
2. Solve for x: $x - 2 = \sqrt{x}.$
3. Solve for x: $(x + 99999)^2 = x + 100029$.

4. Solve for x: $(2015x-13)^2+4030x=50$.
5. Solve for x: $(x^2+3x+4)^2+2x^2+6x=40$.
6. Solve for all real x: $(x^3+3x^2+3x+1)^4-(x^2+2x+1)^3=90$. (Hint: Factor each individual term.)
7. Solve for x: $(3x+5)^2+3x+8=0$.
8. The cube root of a real number equals the number taken to the 2/3th power minus thirty. What is that number?
9. Solve for x: $(\sqrt{x}-3)^2+8\sqrt{x}=41$.
10. Solve for x: $(x-\sqrt{x}+6)^2+5x-5\sqrt{x}=\ -24$.

Sometimes you want to substitute variables for numbers.

Challenge. Compute, without using a calculator, the value of $\frac{1^4+2014^4+2015^4}{1^2+2014^2+2015^2}$. *(Source: British Mathematical Olympiad)*

Notes 2.3 Name: ____________________

Special Algebraic Manipulations, Part 2

<u>Vieta's Formula</u>

Example. Expand (x - a)(x - b).

Solution. $x^2-(a+b)x+ab$.

1. Expand $a(x-x_1)(x-x_2)$.
2. Expand $a(x-x_1)(x-x_2)(x-x_3)$.
3. Compute the roots and their sum and product in:
    a. $x^2+8x+15=0$
    b. $x^2-5x-6=0$
    c. $2x^2+7x+6=0$
4. Prove that the sum of the roots of $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$ is $-\frac{a_{n-1}}{a_n}$.
5. Prove that if r_1, r_2, r_3 are the roots of $ax^3+bx^2+cx+d=0$, then $r_1r_2+r_2r_3+r_3r_1=\frac{c}{a}$. Find an expression for $r_1r_2r_3$.
6. Prove that the sum of the coefficients of a polynomial P is P(1).
7. Let r_1, r_2, r_3 be the roots of $3x^3+8x^2-9x+4=0$. Find $(2-r_1)(2-r_2)(2-r_3)$.
8. Compute the sum of the reciprocals of the roots of $x^2+3x-6=0$.
9. Compute the sum of the squares of the roots of $x^2+10x-55=0$. (Hint: use hint for problem 10)
10. Compute the sum of the cubes of the roots of $x^2+x-1=0$. (Hint: if r is a root of the equation, then r^2 = 1 - r. What is r^3?)
11. (a) Simplify $(x+y+z)^2$.
    (b) Find the sum of the squares of the roots of $x^3+x^2-2x-1=0$.

<u>Arithmetic and Geometric Sequences</u>

Any arithmetic sequence with start term *a* and common difference *d* can be expressed as *a, a+d, a+2d, a+3d, ...*. Any geometric sequence with start term *a* and common ratio *r* can be expressed as *a, a+r, a+2r, a+3r, ...*

1. Prove that each term is the average of its two neighbors in an arithmetic sequence.
2. Determine the sum of the first n terms of an arithmetic sequence in terms of a, d, n.
3. Determine the sum of the first n terms of a geometric sequence in terms of a, r, n. (Hint: let this sum be S. What is r * S? Does it look similar to S?)
4. Determine the sum of all terms of an infinite geometric sequence in terms of a, r.
5. 3 + 6 + 12 + ... + 192 =
6. 3 + 6 + 9 + ... + 102 =
7. 1.5 + 0.75 + 0.375 + ... =

8. A stack of cans is formed by placing 20 cans at the bottom row, 4 on the top row, and such that each row has 2 fewer cans than that of the row below it. How many cans are in this stack?
9. 26 + 22 + 18 + ... - 198 =
10. Find the sum of the first 20 even integers not divisible by 3. (Hint: sum all even integers, then subtract those even integers that are divisible by 3.)
11. I chop a string into thirds and discard one of the thirds. Then, I chop the remaining string also into thirds, and I discard one of the thirds. If I repeat this process infinitely many times, how much string did I discard?
12. Given 1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6, find the sum of the first 20 odd squares. (Hint: we neglected the even squares, which when all divided by 4 leave...?)

<u>Repeated Radicals and Continued Fractions</u>

Example: Compute $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$.

Solution. Letting the entire quantity be x, we have $x = \sqrt{2+x}$. Hence $x^2 = 2+x$, or $x^2 - x - 2 = 0$, so $(x-2)(x+1)=0$. Because x > 0, we have x = 2, which is our answer.

1. $1+\frac{1}{1+\frac{1}{1+\ldots}} =$
2. $\left(\sqrt{2\sqrt{3\sqrt{5\sqrt{2\sqrt{3\sqrt{5\sqrt{2\ldots}}}}}}}\right)^7 =$
3. A sequence a_n is defined recursively by $a_0 = 2014$ and $a_{n+1}{}^2 = 2a_n - 1$. Given that a_n eventually converges to a value N, find N.
4. $2-\frac{3}{6-\frac{3}{2-\frac{3}{6-\ldots}}} =$
5. $3.5+\frac{1}{\sqrt{3.5+\frac{1}{\sqrt{3.5+\frac{1}{\sqrt{3.5+\ldots}}}}}} =$
6. $2+\frac{3}{2+\frac{4}{2+\frac{3}{2+\frac{4}{2+\ldots}}}} =$
7. $\sqrt[4]{14+\sqrt[4]{14+\sqrt[4]{14+\ldots}}} =$

Notes 2.4 Name: ___________________

Inequalities and Optimization

Inequalities

If $a > b$ and $b > c$, then $a > c$.

If $a > b$ and $c > 0$, then $ac > bc$.

If $a > b$ and $c > d$, then $a + c > b + d$.

WARNING!!! If $a > b$ and $\mathbf{c < 0}$, then $ac < bc$.

How to solve linear inequalities:

1. Isolate the variable term and the constant term.
2. Divide both sides by the coefficient of the variable term. Remember to change the inequality if necessary.

Example. If $3x + 7 > 19$, find the least possible integral value of x.

Solution. $3x > 12$, so $x > 4$ and 5 is our answer.

Example 2. Solve: $-5x - 7 > 23$.

Solution. $-5x > 30$, so $x < -6$.

Exercise Set A. *(Solve the inequalities for x.)*

1. $-3x + 36 > -2x - 43$
2. $2x + 30 < -x + 46$
3. $5x + 65 > -2x + 22$
4. $80 < 3x + 67$
5. $2x - 32 \geq -4x - 7$
6. $-x + 50 > -3x + 80$
7. $-5x + 91 > 2x - 94$
8. $-4x - 9 < 5x + 50$
9. $-x - 5 < -3x - 34$
10. $-4x + 82 > -3x + 77$
11. $x - 76 > 5x + 51$
12. $-3x - 11 < 2x - 4$
13. $5x + 41 < -x - 50$
14. $-3x + 32 \geq x - 74$
15. $x + 30 < 4x - 47$
16. $4x - 19 < -4x + 36$
17. $-x + 96 \geq -2x - 56$
18. $-2x + 30 < -4x - 28$
19. $x + 83 > -x + 73$
20. $5x - 93 \leq 2x - 49$

Quadratic Inequalities

To solve quadratic inequalities, you:

1. Move everything to one side.
2. Factor.
3. Consider the signs of each term for different values of x.

If you want to find the maximum or minimum of a quadratic expression, you:

1. Identify the vertex x = -b/2a.
2. Plug in x.

Example 3. Solve for x: (x - 5)(2x - 3) > 0.
Solution. x > 5 or x < 3/2, because these choices of x will make each term either both positive or both negative.

Example 4. Solve for x: (x - 5)(2x - 3) ≤ 0.
Solution. 3/2 < x < 5.

Example 5. Solve for x: x^2 + 2x + 1 > 0.
Solution. Factoring yields (x + 1)^2 > 0. Because a perfect square is always non-negative, our answer is x < -1 or x > 1.

Example 6. What is the maximum possible value of x(6-x)?
Solution. This is $-x^2 + 6x$, so vertex is at -6/(-2) = 3. Plug in x = 3 to get 9.

Exercise Set B. *(Solve the inequalities for x.)*
1. 2x^2 - 14x + 20 > 0
2. 2x^2 + 11x + 15 < 0
3. 3x^2 + 25x + 42 > 0
4. 3x^2 - 3x - 6 < 0
5. 9x^2 - 12x - 12 > 0
6. 4x^2 + 4x - 35 > 0
7. 6x^2 + 13x + 6 < 0
8. 3x^2 + 2x - 5 < 0
9. 9x^2 + 15x - 6 > 0
10. 2x^2 - 3x + 1 > 0
11. 2x^2 - 10x + 12 < 0
12. 6x^2 - 18x + 12 < 0
13. 9x^2 + 15x - 6 > 0
14. 6x^2 + 2x - 20 < 0
15. 6x^2 - 15x - 21 > 0
16. 9x^2 + 9x - 18 < 0
17. 2x^2 - 12x - 14 < 0
18. 9x^2 - 21x + 10 > -2
19. x^2 - x < 2

20. 3x^2 + 8x - 15 > 1

Exercise Set C. Find the maximum or minimum of the quadratic expression, and identify whether it is a maximum or minimum.

1. $x^2 - 8x + 8$
2. $-x^2 + 9x - 9$
3. $x(12 - x) + 4$
4. $x(x - 6) + 18$
5. $3x^2 - 6x + 4$
6. $-8x^2 + 4x + 9$
7. $3x(18 - 5x)$
8. $2x(8 - 3x) + 9$
9. $3(x - 2)^2 + 8$
10. $-4(x - 2)(2x - 5) + 7$

Word problems involving maximum and minimum should always be converted to an algebraic form.

Example 7. A farmer has 4 m of rope to create a yard. What is the maximum possible area of his yard?

Solution. Let the dimensions of his yard be x and 2-x. Then the area is x(2 - x). Consider the parabola f(x) = -x^2 + 2x. Its vertex is (-b/2a, f(-b/2a)) = (1, 1), which is the parabola's maximum point. Thus, 1 is our answer.

Alternate Solution. By the AM-GM Inequality, x(2 - x) ≤ ((x + 2 - x)/2)^2 = 1, which is our answer. (Equality holds for x = 1.)

Example 8. If x > 3, prove that x^2 - 2x - 3 > 0.

Solution. Factoring gives (x - 3)(x + 1) > 0, which is true because both terms are positive.

Exercise Set D.

1. A farmer has 40 m of rope to create a yard. What is the maximum possible area of the yard?
2. A farmer has a rectangular yard with area 36. What is the minimum possible value of the perimeter?
3. Prove that for any positive integer greater than 3, the square of the integer exceeds twice the integer by at least 1.
4. Prove that x^2 + 3x + 5 > 0 for all real x. (Hint: Complete the square.)
5. Prove that ax^2 + bx + c > 0 for all real x if and only if b^2 - 4ac > 0.
6. A collection of marbles has blue and red marbles. Initially there are 10 red marbles and some blue marbles. After 5 blue marbles are added, the probability of a blue marble being drawn is greater than 90%. What is the fewest number of blue marbles present initially?

Notes 2.5 Name: _________________
Number Sense

*Host: "Problem. Find (-2016) * (-2015) * (-2014) * ... * (2014) * (2015) * (2016)."*
*Contestant: "*buzz* It's zero!" *Applause**
*Defeated opponent: "How did you do that so quickly? I was still evaluating (-2016) * (-2015)!"*
Contestant: "There's a zero, dude."

Often, problem-solving is not about memorizing a certain set of guidelines and trying all of them on a problem. Sometimes you will have to do something creative to save yourself a lot of time and effort.

Example 1. Compute 12 + 34 + 66 + 88.
Solution 1. 46 + 154 = 200.
Solution 2. 100 + 100 = 200.

Simple "common sense" tricks like these can simplify your work and reduce possibility for mistakes.

Example 2. Evaluate $\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \frac{4}{6} \cdot \frac{5}{7}$.
Solution 1. It is $\frac{120}{2520} = \frac{1}{21}$.
Solution 2. Use cancellation to arrive at $\frac{2}{6 \cdot 7} = \frac{1}{21}$.

Even the art of simplifying fractions can be simplified. For instance, to simplify 120/2520, you first cancel the zeroes to get 12/252, and then divide out by 2 and 3 (or 6) to get 2/42 = 1/21.

Other tricks may be more hidden:

Example 3. Find $1 \cdot 1! + 2 \cdot 2! + \cdots + 7 \cdot 7!$.
Solution 1. Evaluate each term separately and add to get 40319.
Solution 2. Note that 1 * 1! = 2! - 1, 1 * 1! + 2 * 2! = 3! - 1, which holds in general. The answer is 8! - 1 = 40319.

However, the biggest trick of all may be just mental math.
Example 4. Evaluate $29 \cdot 33$.
Solution. 30 * 33 - 33 = 990 - 33 = 957.

Intuition helps a lot too.

Example 5. The sum of the first three terms of an arithmetic sequence is 90 less than the sum of the next three terms. What is the difference between the sum of the first five terms and the sum of the next five terms?
Solution. We characterize the sequence as a, a+d, a+2d, a+3d, ... Now we realize that we can allow a = 0 (because intuitively the first term doesn't matter in the problem), so the equation is 15d = 90 + 6d, so d = 10. Now our desired difference is

$$(6+7+8+9+10)\cdot d-(1+2+3+4+5)\cdot d.$$

Intuitively 6 + 7 + 8 + 9 + 10 = 8 * 5 = 40, so our difference is 25d = 250. Another way of doing this is noticing that (6 - 1) = 5, (7 - 2) = 5, ..., (10 - 5) = 5, so we have 5 times 5d, or 25d = 250.

Estimation is a general trick to see if your answer is correct or not.
Example 6. Is 89 * 216 = 17894?
Solution 1. 90 * 216 = 18000 + 1400 = 19400, so 17894 seems too small.
Solution 2. Do mod 9 trick, (-1) * 0 = 2 (mod 9) is false. (We computed mod 9 by sum of digits mod 9.)

Distribution and factoring can help a lot:
Example 7. Find $82^2+82*19.$
Solution. It is 82 * 101 = 8282.

Finally, algebra can help in unexpected ways.
Example 8. Determine 99 * 99.
Solution. Let x = 100. Then we have
$(x-1)^2=x^2-2x+1=10000-200+1=9801.$

Useful tip: Memorize the following quantities:

A. The first 50 squares.
B. How to get the next 50 squares from knowledge of the first 50 squares. (Hint: 67 = 50 + 17)
C. The first 12 powers of 2.
D. The first 10 factorials.
E. Prime factorization of 2016 and 2017.

Problems.

1. 23 + 89 + 77
2. 108 - 39
3. 277 - 199
4. 28 * 30
5. 29 * 107
6. 99 * 101
7. 106^2-104^2
8. 83 * 83 + 2 * 83 * 7 + 50

9. 207 * 208
10. 28 * 28
11. Compute all numbers in A.
12. 84 * 84
13. Compute all numbers in B, C, D, E.
14. 40 * 78 - 20 * 155
15. 15 * 32
16. Given that 26244 is a perfect square, find its square root. (Hint: 25600 is a perfect square, and sqrt(26244) > sqrt(25600), so you can find leading digits of sqrt(26244)....)
17. Find the square root of 146689.
18. Using (10a + b)(10c + d) = 100ac + 10(ad + bc) + bd, compute 83 * 34.
19. 98 * 102 + 9605
20. $2^{16} - 2^{15} - 2^{14} - 2^{13} - 2^{12} - 2^{11} - 2^{10}$

CHAPTER THREE

COUNTING & PROBABILITY

Notes 3.0 Name: ____________________

Basics

1. There are n numbers in the list 1, 2, 3, ..., n.

With some cleverness, one can count harder lists like the number of three-digit odd squares. One observes the least odd square is 11^2 and the greatest odd square is 31^2, so we want the number of numbers in 11, 13, 15, ..., 31. We subtract 9 and divide by 2 to each element, resulting in 1, 2, 3, ..., 11. There are 11 3-digit odd squares. Similar reasoning can determine the number of odd multiples of 3 between 1000 and 2520.

2. If you have *m* choices for A and *n* choices for B, then you have *mn* choices for A and B.

(Example: if you have 4 types of shirts and 3 types of pants, then you have 12 types of outfits consisting of 1 shirt and 1 pants.)

3. If you have *m* choices for A and *n* choices for B (and A, B don't overlap), then you have *m* + *n* choices for A or B.

(Example: if you have 4 types of green shirts and 3 types of gray shirts, then you have 7 types of green or gray shirts.)

4. Sometimes you have to divide for overcounting. For instance, the number of ways to rearrange the letters of ALPHA should be 5 * 4 * 3 * 2 * 1 = 120, but in fact the two As are the same, so there are only 60 rearrangements.

5. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the number of ways to choose r members from a group of n elements, where order doesn't matter.

What is more interesting about counting is combining different types of counting to obtain the desired result.

Notes 3.1 Name: ______________________

3 Cs of Counting

3 Cs of Counting: **Casework, Complementary Counting,** and **Constructive Counting.**

Casework is the using of cases to solve a problem. The most important thing is to be accurate and careful about listing cases.

Example 1.0. How many integers less than 1000 do not contain any 0?
Solution. We casework on the number of digits of the integer.
Case 1: 3 digits. Then clearly we have 9 * 9 * 9 = 729 possibilities for this case.
Case 2: 2 digits. This gives 9 * 9 = 81 possibilities.
Case 3: 1 digit. This gives 9 possibilities.
Summing gives a total of 819 total integers.

Exercise 1.1. How many integers less than 1000 do not contain any 2?

Exercise 1.2. How many paths on lattice points from (0, 0) to (5, 5) (going only up and to the right) are there that do not pass through (1, 3), (2, 2), or (4, 2)? (Hint: Use Pascal Counting. Ask if you do not know what it is.)

Exercise 1.3. (Sycamore Ridge Diagnostic Test) The probability it will rain tomorrow in Sunnyville is 30%. The probability it will rain the day after tomorrow in Sunnyville is 40%. However, if it rains tomorrow in Sunnyville, then the probability it will rain the day after of tomorrow will increase to 60%. What is the probability it will rain the day after tomorrow in Sunnyville? Express your answer as a simplified fraction.

Exercise 1.4. Color the vertices of a square in four colors (red, white, blue, green) such that no two adjacent vertices have the same color. How many colorings are possible? (Rotations count as distinct colorings.)

Exercise 1.5. How many sequences of five coin flips do not contain two consecutive heads or three consecutive tails?

Exercise 1.6. I draw 2 cards out of a standard deck of 52 cards without replacement. What is the probability that I draw an ace and then a spade?

Complementary Counting is the counting of the opposite of what you want to count.

Example 2.0. I flip a coin 10 times. What is the probability that at least two flips are heads?
Solution. We count the number of ways to flip only zero or one head, which is basic casework! There are 10C0 + 10C1 = 1 + 10 = 11 ways to flip zero or one head. Thus, the probability of

flipping one or zero heads is 11/1024, so the probability of flipping two or more heads is 1013/1024.

Exercise 2.1. I flip a coin 8 times. What is the probability that at least three flips are heads?

Exercise 2.2. I win a game with probability 80%. I play three games. What is the probability that I win at least one game out of these three, as a percent?

Exercise 2.3. How many 3-digit integers contain at least one zero in their decimal representation?

Exercise 2.4. In the Paris Climate Change talks, there are 20 delegates seated in a line, among them Presidents Obama and Putin. If they do not wish to sit next to each other, how many ways are there to seat the 20 distinct delegates? (Hint: count the complement, and treat Obama and Putin as an OP or PO block.)

Exercise 2.5. How many even 2-digit integers are not multiples of 5?

Constructive Counting is the counting of specific elements of what you want to count, and then multiplying everything together.

Example 3.0. How many 4-digit integers are there?
Solution. There are 9 digits for the thousands digit, and 10 for each of the other digits. Thus, there are 9 * 10 * 10 * 10 = 9000 4-digit integers.

Exercise 3.1. Pies at a certain bakery can have 5 types of toppings, 4 types of crust, and 3 types of flavor. How many possible pies are there?

Exercise 3.2. A pizza can have up to one each of 10 different toppings. Prove that there are 1024 different possible pizzas that can be made.

Remember we may have to divide by a certain number for the overcount.

Example 3.3. There are 10 delegates at a conference. Each delegate shakes hands with exactly 5 of the others. How many handshakes took place?
Solution. There are 10 * 5 = 50 total delegate-handshakes. However, each delegate is part of two handshakes, so we divide by 2. The answer is 25.

Exercise 3.4. In terms of n, how many diagonals are there of a convex n-gon?

Exercise 3.5. How many triangles are formed by the vertices of a 20-gon such that no two vertices are adjacent? (Hint: choose a starting vertex and then choose the other two vertices. Watch out for overcount!)

Notes 3.2 Name: ______________________

1-1 Correspondence, Expected Value

Sometimes you can count things by mapping possibilities to a well-known object.

Example 0. How many ways are there to go from (1, 5) to (8, 9) going 1 unit up or to the right each move?

Solution. First we move everything down (1, 5), so we want to go from (0, 0) to (7, 4). Now notice we need 4 ups and 7 rights, so $\binom{11}{4}$ ways to organize them.

Exercise 1. Chris walks from home (0, 0) to school (20, 20) each day moving 4 blocks to the right and 5 blocks upwards each move. However, he may not pass through the house of a notorious bully at (12, 10). How many possible walks does Chris have?

Sticks and Stones.

Example 4.0. I have 10 candies to give to 4 students, such that everyone must receive at least one candy. How many ways can I do so?

Solution. Consider 10 stones. These 10 stones leave 9 gaps. If we place 3 sticks in 3 of these gaps, then we will have a desired distribution. For example,

x x | x x | x x x x | x x

means that the first students receives 2, the second 2, the third 4, and the forth 2. Therefore, our answer is $\binom{9}{3}$ = 9 * 8 * 7 / 6 = 84.

Example 4.1. How many ways can I give the 10 candies to 4 students if everyone must receive at least 2?

Solution. First, give everyone one candy. Then we have the previous problem, except with 6 candies; similarly, we find the answer is $\binom{5}{3}$ = 10.

Example 4.2. How many ways can I give the 10 candies to 4 students without restrictions?

Solution. Distribute the 10 stones and 3 sticks to provide a valid arrangement. For example,

x x x x x x x x x x | | |

can become

x x | x x x | | x x x x x,

which corresponds to 2-3-0-5. The answer is $\binom{13}{3}$ = 13 * 12 * 11 / 6 = 286.

Exercise 4.3. How many ways can I give *m* $1 dollar bills to *n* people without restrictions?

Exercise 4.4. How many ways can I give *m* $1 dollar bills to *n* people if everyone must receive at least *k* of them?

Exercise 4.5. How many ways can I distribute 12 candies to 5 children if two of the children are twins and insist on receiving the same, positive amount? (Hint: Use casework.)

Exercise 4.6.

(a) How many ways can I make $999 with $1 and $10 dollar bills?

(b) How many ways can I make $999 with $1, $10, and $100 dollar bills?

Exercise 4.7.

(a) Compute the number of positive integral solutions to $a+b+c+d+e=15$.

(b) Compute the number of non-negative integral solutions to $a+b+c+d \leq 15$. (Hint: How does this relate to part (a)?)

(c) Compute the number of integral solutions to $a+b+c+d=16$ with each of a, b, c, d greater than 2.

Exercise 4.8.
My lottery ticket has five numbers from 1 to 10, not necessarily distinct. If the order of the numbers does not matter, how many possible lottery tickets are there? (Hint: Let the numbers be a ≤ b ≤ c ≤ d ≤ e. What can you say about a, b+1, c+2, d+3, e+4?)

Combinations Problems.
I have a lock whose three dials each display a number between 000 and 999. Call any number displayed on the three dials a *key.*
Exercise 5.0. How many keys have at least one 5?
Exercise 5.1. How many keys have no repeating digits?
Exercise 5.2. How many keys are odd?
Exercise 5.3. How many keys are odd multiples of 5?
Exercise 5.4. How many keys have digits in nondecreasing order? (e.g. 112, 123, 233, 255)
Exercise 5.5. How many keys have at least one 1 and one 5?
Exercise 5.6. How many keys have exactly one 1?
Exercise 5.7. How many keys have sum of digits equal to 7?
Exercise 5.8. How many keys have sum of digits equal to 21?

Expected Value is the average of all possible scores, probability-weighted.

Example 1. I roll a die. What is the expected outcome?
Solution.

Value	1	2	3	4	5	6
Probability	1/6	1/6	1/6	1/6	1/6	1/6

Contribute	1/6	2/6	3/6	4/6	5/6	6/6

The sum of all contributions is 21/6 = 3.5, the expected value.

If an evil hacker changes the 6 on the die to a 5, then the expected value becomes (1 + 2 + 3 + 4 + 5 + 5)/6 = 20/6 = 3.33.... If we subtract all numbers on a regular die by 1, then the expected value is ______________________ = 2.5. Notice that this is exactly 1 less than 3.5!

Intuitively, you can consider each individual object separately while considering expected value.
Theorem of Expected Value. If E(A) is the expected value of event A, then
E(A + B) = E(A) + E(B).

Example 2. I roll 3 dice. What is the expected outcome (of the sum of the rolls)?
Solution. By example 1, the expected value of one die is 3.5. Thus, for three dice, the "average" number on each is 3.5, so the total expected value is 3.5 * 3 = 10.5.

If we subtract all numbers on one of the dice by 1, then the expected value also decreases by 1. If we subtract all numbers on another of the dice by 2, then the expected value of that die decreases by 2. Thus, the expected outcome becomes ____ + ____ + ____ = 7.5.

Exercise 1. I roll 100 dice. What is the expected value of the outcome?

Exercise 2. 7-sided magic dice have numbers 2, 4, 5, 7, 8, 10, 12. What is the expected value of rolling 28 magic dice?

Exercise 3. I roll 35 magic dice, 30 regular dice, and flip 40 coins. A coin has value 6 if it is heads and 1 if it is tails. What is the expected value of the 105 objects?

Fair games have expected value equal to the price of the game. Of course, not all games are fair....

Example 3. Black and Jack play blackjack. It turns out that Black, the dealer, has a 51% of winning blackjack. If he wins $1 from Jack if he wins, how much should he pay Jack when he loses such that the game is fair?
Solution. Let Black pay Jack $x when he loses. Then the expected value of the fair game is zero (because there is no price), so 0 = .51 (1) + .49 (-x). (Black wins $1 when he wins and wins (-x) when he loses.) Thus x = 51/49 = $1.04.

Exercise 4. A lottery ticket costs $1. The chances of winning the $1,000,000 lottery is 1 in 4 million. What is the expected value of a lottery ticket?

Exercise 5. A 36-spoked wheel has 1 slot marked $100, 4 slots marked $10, and 5 slots marked $1. (The rest are marked $0.) You win the money listed on the slot that you spin, and you can spin three times. If the game manager wants to make an expected profit of $1 per game, what price should he charge for people to play it?

Often the difficulty lies in “picking the problem apart.”

Example 4. Ten girls and five boys line up randomly in a line. What is the expected number of places in which a girl is standing next to a boy?

Solution. Consider the first two spots. The probability that they are boy-girl or girl-boy is 5/15 * 10/15 + 10/15 * 5/15 = 4/9. Thus, the expected number of boy-girl combinations in these two spots is 1 * (4/9) + 0 * (5/9) = 4/9. (If we have “success” with probability 4/9, then we add 1 to our girl-boy count.) Because there are 14 total sets of two adjacent spots, the total expected number of boy-girl places is ____ * 14 = 56/9.

Exercise 6. The top 10 members of the PTMS Mathcounts team attend a dinner party but can’t see the name tags on the 10 available seats. Thus they seat themselves randomly. What is the expected number of members who seat themselves in front of their own name tag? (Hint: What is the probability that in seat 1, the person corresponds to that name tag? How about seat 2? 3? ...)

Exercise 7. A random 52-letter word is formed using letters of the alphabet. What is the expected number of Es and Ms in the word?

Exercise 8. Ten coins with radius 1 are placed on a circular table with radius 10. I randomly drop a coin with radius 1 onto the table. What is the expected number of coins my coin will overlap? (Hint: What is “overlapping” in terms of the coins’ centers? Then use geometric probability. Concentrate on one coin at a time.)

Notes 3.3 Name: ____________________
Probability and Geometric Probability

Probability is number of valid outcomes / total number of outcomes. To solve probability problems, you use the same methods as in counting. Shortcuts can also be used.

Example 1. What is the probability that one flips three heads in a row?
Solution 1. There is only one way to flip three heads in a row, but there are 8 outcomes total. The answer is _____.

Solution 2. The probability that the first coin is heads is 1/2, and similarly for the others. Therefore, our desired probability is _______.

Exercise 0. I draw 2 cards (a) with replacement; (b) without replacement; from a deck of 52 cards. What is the probability that the first card is an ace and the second card is a five?

Binomial Probability is the probability that something happens exactly m out of n times.

Example 2. What is the probability that out of 8 coin flips, exactly 3 are heads?
Solution. There are 8C3 possible sequences with exactly 3 out of 8 being heads, and the probability that that sequence occurs is $(1/2)^3(1-1/2)^5 = 1/256$. Hence the answer is __________________.

Complementary Probability is based on the principle of Complementary Counting.

Example 3. It rains with probability 60%. What is the probability that it does not rain?
Solution. 100% - 60% = ____.

Exercise 1. A biased coin flips heads with probability 1/3 and tails with probability 2/3. What is the probability that out of 7 coin flips, exactly 4 are tails?

Exercise 2. It rains with probability 2/3 in Mathlandia. What is the probability that out of 10 days, at least 8 are rainy days?

Exercise 3. It snows with probability n in Mathlandia. What is the probability that out of m days, exactly k are not snowy?

Exercise 4. I win a lottery with probability 1/100. What is the probability that I play 100 lottery games and win at least one of them?

Geometric Probability is probability of geometry. Obviously, the probability that a point in a region P is picked given a total region R is area(P) / area(R).

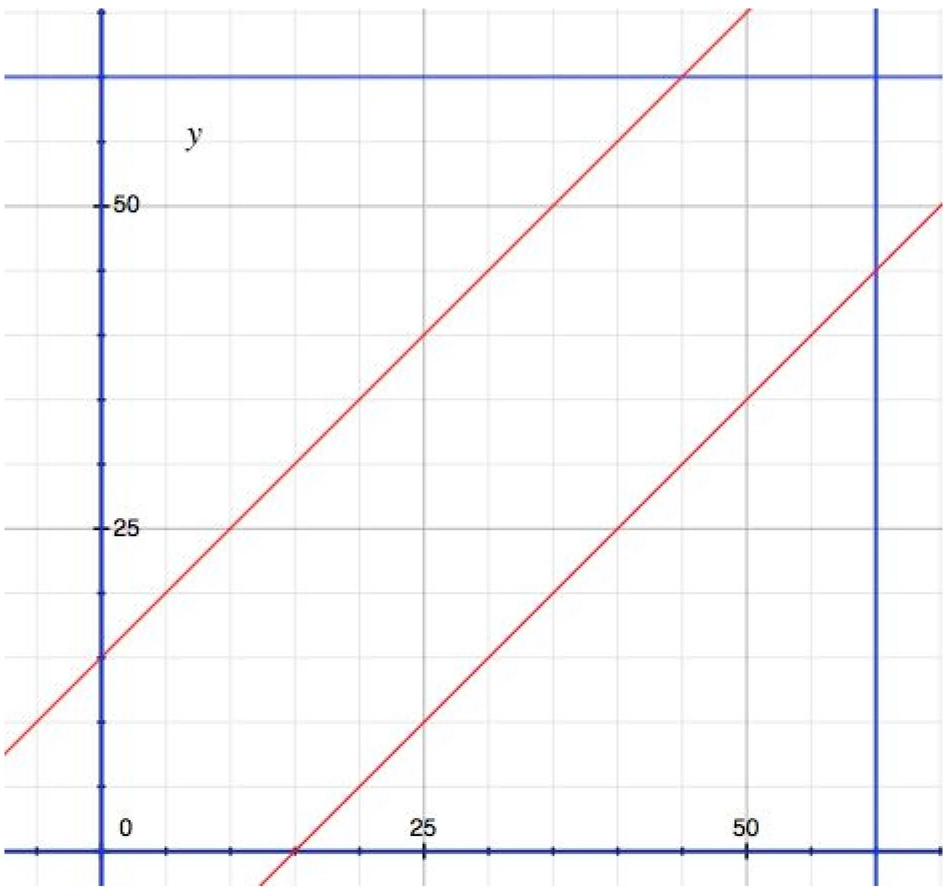
Example 4. I throw a dart onto a target with three rings of radius 1 each. What is the probability that I hit the outermost ring?

Solution. The area of the outermost ring is 9π - 4π = 5π. The area of the circle is 9π. Therefore, the answer is __________.

Usually you use geometric probability when there are infinitely many possible states.

Example 5. David and I arrange to meet at the library, but we forgot to agree on a time. We arrive at a random time between 3:00 and 4:00 PM. Each of us will wait 15 minutes for the other person to come before leaving. What is the probability that we meet?

Solution. Draw a coordinate system. The origin is modeled by (3:00, 3:00), and 4:00 is at a distance 60 (minutes) from the origin. The total region is the 60x60 square, and a point (x, y) represents (time David arrives, time I arrive).



The "good" region is bounded above and below by the lines y = x + 15 and y = x - 15, respectively. But this region is the entire square minus two isosceles right triangles with leg 45. Thus, the desired probability is _____________________.

Exercise 5. I randomly drop a metal coin with radius 1 onto a square with side length 5. Find the probability that the coin will overlap with the exterior of the square.

Exercise 6. David and I arrange to meet at the movie theater, but we forgot to agree on a time. We arrive at a random time between 7:00 and 8:00 PM. I will wait 15 minutes for David to come, while David will wait 20 minutes for me to come before leaving. What is the probability that we meet?

Exercise 7. David and I arrange to meet at the movie theater, but we forgot to agree on a time. I arrive at a random time between 7:00 and 8:30 PM, and David arrives at a random time between 7:30 and 9:00 PM. I will wait 20 minutes for David to come, while David will wait 30 minutes for me to come before leaving. What is the probability that we meet?

Notes 3.4 Name: ___________________

PIE

The **Principle of Inclusion-Exclusion** (PIE) counts the number of items in the union of some sets.

Example 1. I have 10 male dogs and 20 female dogs. If for some odd reason 7 of the dogs are both male and female, how many dogs do I have total? (Please do not laugh.)
Solution. First, we add 10 and 20 to get 30 dogs total. However, we have to subtract the overcount that occurs because we counted each male/female dog in both categories. Thus, the answer is 10 + 20 - 7 = 23.

Problem Set A.

1. Your mother screams at you 55 times each day and yells at you 47 times each day, and all this occurs during 86 incidents (in which she either screams or yells at you or both). How many times each day does your mother both yell and scream at you at the same time?
2. During a computer game that lasts 60 minutes, you feel happy for half of the time and you feel sad for two-thirds of the time. You feel neither emotion for 6 minutes (while the game is loading). How many minutes are you both happy and sad?
3. In a group of 50 aliens, 34 want to invade Earth, 18 want to invade Mars, and 11 want to do both. How many aliens want to do neither?
4. In a group of 60 aliens, 46 want to invade Jupiter, 26 want to invade Saturn, and 10 want to do neither. How many aliens want to do both?
5. A teacher teaches a class of 70 students that know either math or science (or both). 16 students do not know math, and 27 students do not know science. What is the least number of students that know both? (Hint: How many students know math? science?)

Overcounting and undercounting gets more complicated with three sets involved.
Example 2. Sycamore Ridge has 900 students that like either cats, dogs, or parrots. 300 like cats, 400 like dogs, and 340 like cats. 60 like both cats and dogs, 70 like both cats and parrots, and 100 like both dogs and parrots. How many students like all three?
Solution. Let us count a different way the number of students that like either cats, dogs, or parrots. Consider 300 + 400 + 340 = 1040. This overcounts the students who like two of cats, dogs, or parrots, and so we subtract 60 + 70 + 100 = 230. However, now we have undercounted the number of students who like all three (they were originally counted 3 times, and now we subtracted them 3 times). Hence, we add back x, the number that like all three. Our equation becomes 900 = 1040 - 230 + x, or x = 90.

In general, be careful of what you are counting and overcounting.
Example 3. In a group of 60 musicians that know how to play at least two of the violin, cello, and piano, 23 know how to play the violin and cello, 26 know how to play the violin and piano,

and 17 know how to play the cello and piano. How many musicians know how to play all three instruments?
Solution. Notice that 23 + 26 + 17 = 66 overcounts twice the number of people able to play all three instruments. Thus, 66 - 2x = 60, so x = 3.

Problem Set B.

1. In Mathlandia, citizens are of three different genders–positive, negative, and neutral. Citizens must have at least one gender, but they can have multiple genders. 160 are positive, 240 are negative, and 180 are neutral. 80 are positive and negative, 40 are positive and neutral, and 70 are negative and neutral. 30 are of all three genders. How many citizens are there in Mathlandia?
2. In a group of candy-eaters, 40 people eat candy types A, B, C; 30 people eat candy types A, C, D; 35 people eat candy types A, B, D; 20 people eat candy types B, C, D; and 10 people eat all four types of candy. How many people eat at least three types of candy?
3. Torrey Pines has 560 students that like either cats, dogs, or parrots. 317 like cats, 399 like dogs, and 287 like parrots. 182 like both cats and dogs, 211 like both cats and parrots, and 69 like all three. How many like both dogs and parrots?
4. In a group of 70 people, only 10 are not good at at least two of math, science, and English. Also, 23 are good at math and science, 26 are good at math and English, and 8 are good at math, science, and English. How many people are good at science and English?
5. Given that $|A|$ represents the number of elements in a set, $A \cap B$ represents the intersection, and $A \cup B$ the union of two sets, express $|A \cup B \cup C|$ in terms of $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |A \cap C|, |A \cap B \cap C|$.

Additional practice.

1. In a standard 52-card deck, how many cards are fours or hearts?
2. How many 2-digit integers are multiples of 2 or 3?
3. How many 3-digit integers are multiples of 2, 3, or 5?

Notes 3.5 Name: ____________________

Pigeonhole Principle

Intuitively, if you put 8 pigeons in 7 holes, then two pigeons will be unhappy because they share the same hole.

1. Suppose we try to put 100 pigeons into 7 holes. At least how many pigeons will share a common hole?

The hardest part of a question is deciding what are the pigeons and what are the holes.

2. I have 50 socks each of 100 colors. What is the fewest number of socks I need to draw to guarantee a pair of matching socks?
3. I have 70 each of red, green, and blue balloons. What is the least number of balloons you need to buy from me to ensure that you have 50 balloons each of the same color?
4. Your password has 3 letters, each a letter (uppercase or lowercase) or a digit between 0 and 7. A hacker attempts to figure out your password. What is the greatest number of attempts the hacker needs?
5. What is the maximum number of integers I can choose between 1 and 100 such that
   a. No two are consecutive
   b. No two differ by exactly 30
   c. No two a, b exist such that a - b is divisible by 5
   d. No two a, b exist such that ab(a-b) is divisible by 7
   e. No two differ by at most 30
   f. No three are consecutive
   g. None divides the other?
6. What is the least number of darts on a dartboard of diameter 2 meters such that at least two darts are less than 1 meter apart?

CHAPTER FOUR

NUMBER THEORY

Notes 4.0 Name: ______________________
Basics

There are many types of numbers.

Natural numbers: 1, 2, 3, 4, 5, ...
Whole numbers (non-negative integers): 0, 1, 2, 3, 4, ...
Integers: 0, 1, 2, 3, 4, 5, 6, ... and -1, -2, -3, -4, -5, ... (or 0, ±1, ±2, ±3, ...)
Rational Numbers: All fractions of the form a/b, where a, b are integers (b ≠ 0)
Real Numbers: All numbers that can be expressed as a decimal (e.g. π, e)

Number theory deals with the **integers.**

We say a **divides** b, or a | b, if (b / a) is an integer. For example, 3 | 6 because 6 / 3 = 2. But 3 does not divide 5 because 5 / 3 = 1 R 2. You can use long division to test for divisibility or use the divisibility rules.

For divisibility by:

1. Trivial.
2. Last digit is even.
3. **Sum of digits is divisible by 3.**
4. **Last two digits are divisible by 4.**
5. **Last digit is 0 or 5.**
6. **Divisible by 2 and 3.**
7. Three methods:

(a) 100a + b is divisible by 7 if and only if 2a + b is divisible by 7. For example, 105→ 2 * 1 + 5 = 7 is divisible by 7.
(b) 10a + b is divisible by 7 if and only if a - 2b is divisible by 7. For example, 105 → 10 - 2 * 5 = 0 is divisible by 7.
(c) Just do the long division! For example, 105/7 = 15.

8. **Last three digits are divisible by 8.**
9. **Sum of digits is divisible by 9.**
10. **Last digit is 0.**
11. **Alternating sum of digits is divisible by 11.**

For example, 123456 -> 1 - 2 + 3 - 4 + 5 - 6 = -3 is not divisible by 11.

12. Divisible by 3 and 4.
13. Just do the long division!
14. Divisible by 2 and 7.
15. Divisible by 3 and 5.
16. Last 4 digits divisible by 16.
17. For primes 17 and greater, do the long division!

In general if you need divisibility by *ab* where gcd(a, b) = 1, it suffices to test divisibility by *a* and *b*.

Exercise 101-130. For exercise x, determine whether x is divisible by 2, 3, 7, 8, 9, or 11 using divisibility rules.
Exercise 131. AB982C is a 6-digit number divisible by 15. Find all possible values of A + B.
Exercise 132. 4B6D0 is a 5-digit number divisible by 900. Find all possible values of B.
Exercise 133. What is the divisibility rule for 420?

A number *a* is a **factor** or **divisor** of *b* if a | b. The **factors** of a number *n* are all *m* such that m | n. Usually **we have that m is positive.**

prime - A number with 2 positive factors: 1 and itself. In particular, **1 is not prime.**
For example, 5, 7, and 13 are primes.
prime factorization - The splitting of a number into prime factors and their exponents.
For example, 2014 = 2 * 19 * 53 and 72 = $2^3 * 3^2$.

It is a theorem that each positive integer greater than 1 has a unique prime factorization. Usually that is what you want to do if you have a large number such as 72 or 216.

LCM - The least number that is a multiple of both a and b.
For example, the LCM of 16 and 9 is 144. The LCM of 6 and 15 is 30.
GCD - The greatest number that is a divisor of both a and b.
For example, the GCD of 28 and 16 is 4. The GCD of 144 and 216 is 72.

Notes 4.1 Name: ____________________
Prime Factorization

To test a number for factors, you only need to test primes up to the square root of that number.
Example. Prove 223 is prime.
Proof. We test numbers up to √223 ~ 14. 219 is not divisible by 2, 3, 5, or 11 by our divisibility rules. 7 is easily checked. Finally, 223 / 13 - 17 R 2. Thus, 223 is prime.

To find LCM, you take the larger of the two exponents for each prime in the prime factorization. For GCD, you take the smaller of the two exponents.
Example. Find the LCM and GCD of 12 and 42.
Solution. 12 = 2^2 * 3. 42 = 2 * 3 * 7. Thus, LCM = 2^2 * 3 * 7 = 84. GCD = 2 * 3 = 6.

Notice that LCM(a, b) * GCD(a, b) = ab in that last example. It holds in general (try to see why!)

Exercise 1. Find the LCM and GCD of (a) 120 and 250; (b) 72 and 144; (c) 72 and 192; (d) 24 and 196; (e) 22 and 47; (f) 16 and 88.

Exercise 2. Determine whether the following numbers are prime: (a) 261; (b) 197; (c) 91; (d) 187; (e) 301; (f) 199; (g) 47.

Exercise 3-100. For exercise number x, prime factorize x.

You can rapidly find GCD using the Euclidean Algorithm.
Theorem. *(Euclidean Algorithm Basis)* gcd(a, b) = gcd(a - b, b).
Proof. If k divides a and b, then k must divide a - b as well. Conversely, if k divides a - b and b, then k divides (a - b) + b = a as well. Thus, gcd(a, b) = gcd(a - b, b).

Corollary. gcd(a, b) = gcd(a - kb, b).
Proof. Apply the previous theorem k times.
Example. Find the GCD of 300 and 120.
Solution. gcd(300, 120) = gcd(300 - 2 * 120, 120) = gcd(60, 120) = gcd(60, 60), so the answer is 60.

Exercise 101. Using the Euclidean Algorithm, find the GCD of (a) 150 and 1300; (b) 288 and 1496; (c) 101 and 101000001; (d) 900 and 2790; (e) 1960 and 250; (f) 100 and 1000.

If you have an equation of the form ab = n, then a and b are (not necessarily positive) divisors of n. Then you can casework on the factors of n to explicitly find a and b.

Notes 4.2 Name: ____________________

Factor Analysis

Theorem. *a* divides *b* if for each prime *p*, the exponent of *p* in *a* is less than that in *b*.

Example 1. 6 is a factor of 120 because 6 * 20 = 120, or because 6 = 2 * 3 and 120 = $2^3 * 3 * 5$, and each exponent of a prime in 6 is less than or equal to the corresponding exponent of the prime in 120.

Let $n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ be the prime factorization of n.

Number of Factors: $(e_1+1)(e_2+1)(e_3+1)...(e_k+1)$.
Proof. Clearly a factor must be of the form $p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$, where $0 \le x_i \le e_i$. Thus, each x_i can be one of $e_i + 1$ values, implying the formula by constructive counting.

Product of Factors: We take factors in pairs (if x is a factor then so is n/x), and so the product of factors of n is $n^{(\#\, factors\ of\ n)/2}$.

Sum of Factors: Observe this is given by
$(1+p_1+p_1^2+p_1^3+...+p_1^{e_1})(1+p_2+p_2^2+p_2^3+...+p_2^{e_2})...(1+p_k+p_k^2+p_k^3+...+p_k^{e_k})$,
because on multiplying, each factor of n is represented exactly once.

Exercise 1. What is the number of square factors of $2^{200}3^{199}5^{198}$?

Example 2. What is the sum of the squares of factors of 144?
Solution. 144 = $2^4 3^2$. Consider $(1+2^2+2^4+2^6+2^8)(1+3^2+3^4)$. If we multiply this out, each square of a factor of 144 is represented exactly once in the sum. Thus, our answer is 341 * 91 = 31031.

Exercise 2. What is the sum of the cubes of factors of 72?

Exercise 3. Use the geometric series formula to write the sum of factors formula in a different way. *(7)*

Exercise 4-40. For exercise number x, compute the number and sum of the factors of x. *(Hint: Use your factorizations from Notes 4.1.)*

Exercise 41. What is the prime factorization of the product of the factors of (a) 72 (b) 144 (c) 112 (d) 91 (e) 1000 (f) 160?

Exercise 42. Find the sum of the sum of the sum of the factors of 36.

Exercise 43. Define a *perfect number* to be a number equal to the sum of its proper factors (factors not including itself). For example, 6 is perfect because 1 + 2 + 3 = 6. Define an *abundant number* to be a number *n* whose sum of proper factors is greater than *n*; define a *deficient number* to be a number *n* whose sum of proper factors is less than *n*.

a. Show that 28 is a perfect number.
b. Determine whether 7, 13, 26, 18, 30, 25 are perfect, abundant, or deficient.
c. Prove that all prime numbers are deficient.
d. Show that all numbers of the form $2^n(2^{n+1}-1)$, where $2^{n+1}-1$ is prime, are perfect numbers.

Notes 4.3 Name: ___________________

Algebraic Number Theory: The Collision of Worlds

Factorization is key. Sometimes you will have to rewrite equations so that they factor.

Divisibility also helps. For instance, if a = 2b + 4k, then you know a = 2m for some m, so m = b + 2k.

Part 0. The Army of Counting

Count the number of positive integer solutions to:

1. a + 2b = 50
2. 3a + 2b = 50
3. a + b ≤ 50
4. 3a + 2b ≤ 50
5. 7a + b = 1000
6. a + 2b + 10c = 100
7. Ways to make change in pennies, nickels, dimes, and quarters for a dollar

$(x+1)(y+1) = xy + x + y + 1$.

Part 1. The Army of Algebra

9. Factor $8xy + 4x + 4y + 2$.
10. Factor $12xy + 8x + 3y + 2$.
11. Factor $x^2 - y^2$.
12. Factor $x^2 - 4y^2$.

$xy = 18$ means that x, y are divisors of 18.

Part 2. The Army of Number Theory

Find all positive integer solutions to:

13. $(x+1)(y+1) = 18$.
14. $(x+y)(x-y) = 24$.
15. $(x+2)(x-y) = 22.$
16. $(2x+y)(x-y) = 36$.
17. $(x+y)(x+z)(z) = 28$.
18. $xyz = 48$.

Part 3. The Clash of Forces

Find the number of positive integral solutions to:

19. $xy + x + y + 1 = 48$.
20. $4xy + 2x + 2y + 1 = 36$.
21. $xy + x + y = 799$.
22. $6xy + 2x + 3y = 899$.
23. $x^2 = 144 + y^2$

24. $x^2 = 289 + y^2 + 2y$.
25. $x^2 + 2x = 576 + y^2 - 2y$.

<u>Part 4.</u> The Final Blows

26. *n* is constant and odd. How many solutions does $x^{-1} + y^{-1} = n^{-1}$ have in positive integers, in terms of the number of factors of *n*^2?
27. *b* is constant and odd. How many solutions does $a^2 + b^2 = c^2$ have in positive integers, in terms of the number of factors of *b*^2?

<u>Part 5.</u> Digits

28. Solve 10a + b = 86, where a, b are digits.
29. A 2-digit number plus its reverse is 110. Find all possibilities for the number.
30. How many 6-digit and 5-digit palindromes are there?
31. The positive integers from 1 to 10000 are written in a line, such that each odd integer is written twice. (The string looks like 112334556778...) What is the 10000th digit written?
32. Say a number has *life* if it is a 4-digit palindrome and is divisible by 7. How many numbers have life? (Hint: look mod 7)
33. For which *a* is (1/a) a repeating decimal?

Notes 4.4 Name: ____________________

Modular Arithmetic

NOTE: Show all work and prove all results for all problems. Failure to do so will result in the deduction of points and a possible victory for the other pair.

Define $a \equiv b \ (mod\ c)$ if $a - b$ is divisible by c, or equivalently if $a - b = kc$ for some integral *k*.

Example 1. $2 \equiv 101 \ (mod\ 9)$ but not mod 13.

Then if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then these facts are obvious:

1. $a + c \equiv b + d \ (mod\ n)$.
2. $ac \equiv bd \ (mod\ n)$.
3. $a^e \equiv b^e \ (mod\ n)$ for integral *e*.
4. $a/e \equiv b/e \ (mod\ n/gcd(n, e))$ for integral *e*.

Remember that -1 is defined in modular arithmetic. This allows fast computations of some modular residues.

Example 2. Find the remainder when 3^{100} is divided by 4.

Solution. $3^{100} \equiv (-1)^{100} \equiv 1 \ (mod\ 4)$.

Alternate Solution. $3^{100} \equiv 9^{50} \equiv 1^{50} \equiv 1 \ (mod\ 4)$.

Exercise 5. Find the remainder when 7^{2015} is divided by 50.

Exercise 6. Find the remainder when 7^{1000} is divided by 100. (Hint: What is 7^4?)

Exercise 7. Find the remainder when $9^{200} * 6^{140}$ is divided by 7.

Exercise 8. Find the remainder when $6^{215} + 38^{215} + 370 + 3701^3$ is divided by 37.

If $a \equiv b \ (mod\ m)$ and $a \equiv b \ (mod\ n)$, then $a \equiv b \ (mod\ lcm(m, n))$. This is easily generalized to three modulos or more.

Exercise 9. I have a bag of marbles. Whenever groups of 8 are made with these marbles, 1 marble remains without a group. The same holds for groups of 6 and groups of 10. What is the least number of marbles in the bag?

Exercise 10. Prove that $a^5 \equiv a \ (mod\ 10)$ for any integer *a*. *(Hint: Prove the claim for mod 2 and mod 5. What cases do you have to consider?)*

Thus, $a^b \equiv a^{b \ mod\ 4} \ (mod\ 10)$. Notice that for example $a^{100} \equiv a^4 \ (mod\ 10)$.

Exercise 11. What is the units digit of (a) 3^{2015} (b) 6^{1006} (c) 7^{2992} (d) 132^{132} (e) 2019^{2015}?

Chinese Remainder Theorem: Given that $x \equiv a_i \ (mod\ n_i)$ for $1 \le i \le k$, where all the n_i are relatively prime, then there exists an *N* such that $x \equiv N \ (mod\ n_1 n_2 ... n_k)$.

Example 3. Solve for x: $x \equiv 2 \ (mod\ 3)$, $x \equiv 3 \ (mod\ 5)$.

Solution. The answer must be of the form $x \equiv N \ (mod\ 15)$. To guess *N* (which should be between 0 and 14), we start from the second (more restrictive) condition: x = 3, 8, 13. Of these possible x, only 8 equals 2 (mod 3), and so N = 8. Thus, our answer is $x \equiv 8 \ (mod\ 15)$.

When you have three congruences, you solve any two to yield a new congruence, and use that new congruence and the remaining given congruence to solve the problem.

Exercise 12. David has a bag of marbles that contains fewer than 2000 marbles. Whenever groups of 7 are made with these marbles, 3 remain without a group. Whenever groups of 5 are made, 3 remain without a group. Finally, whenever groups of 3 are made, none remain without a group. What is the greatest number of marbles in David's bag?

Exercise 13. Define the *inverse* of a number *a*, a^{-1} to be the number *b* (mod n) such that $ab \equiv 1 \ (mod\ n)$.

a. Find 2^{-1} and 3^{-1} (mod 5).
b. Does 2^{-1} (mod 6) exist? If so, compute it. If not, explain why not.
c. Let *c* be relatively prime to *a* and *b*. Prove that $a^{-1} \equiv b^{-1} \ (mod\ c)$ if and only if $a \equiv b \ (mod\ c)$.
d. Compute $1/7 + 1/43 \ (mod\ 299)$. *(Hint: Adding fractions is legal.)*
e. Show that $899/3 \equiv 1899/3 \ (mod\ 1000)$, and compute $799/3 \ (mod\ 1000)$.

Exercise 14. A linear congruence is of the form $ax \equiv b \ (mod\ c)$. You can solve it by multiplying both sides by $a^{-1} \ (mod\ c)$.

a. Solve for x: $2x \equiv 5 \ (mod\ 7)$.
b. Solve for x: $3x \equiv 10 \ (mod\ 100)$.
c. Solve for x that satisfy both $2x \equiv 8 \ (mod\ 13)$ and $3x \equiv 5 \ (mod\ 7)$.

Exercise 15. Fermat's Little Theorem: If *p* is a prime and *a* is not divisible by *p*, then $a^{p-1} \equiv 1 \ (mod\ p)$.

a. Compute $2^4,\ 3^4,\ 4^4 \ (mod\ 5)$ without using Fermat's Little Theorem.
b. Compute $2^{2016} \ (mod\ 2017)$, given that 2017 is prime.
c. Compute $2^{4033} \ (mod\ 2017)$.
d. Compute $2^{p^p} - 2 \ (mod\ p)$ for a prime *p*.

General exercises dealing with least common multiple.

16. What is the smallest positive integer that is 1 more than a multiple of the first 6 positive integers?

17. What is the largest 4-digit integer that is 2 less than a multiple of the first 7 positive integers?

18. How many 4-digit integers are 3 more than a multiple of 2, 3, 5, 8, 13?

Chicken McNugget Theorem: Consider the equation $5a + 7b = n$ (*) for some n.

19. Show that (*) has *non-negative integer* solutions for all multiples of 7.
20. Show that (*) has non-negative integer solutions for all positive n = 5 (mod 7).
21. Find the largest n = 1 (mod 7) such that 5a + 7b = n has no solution in non-negative integers.
22. Find the largest n such that 5a + 7b = n has no solution in non-negative integers.
23. Prove that the largest n such that ax + by = n has no solution in non-negative integers x, y is ab - a - b if a, b are relatively prime. What happens if a and b are not relatively prime?

Notes 4.5 Name: ______________________

Counting in Number Theory

Sometimes it is beneficial to use counting techniques to compute stuff in number theory.

Floor function: Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. For instance, $\lfloor 2.2 \rfloor = 2$ and $\lfloor -2.2 \rfloor = -3$.

1. Compute:

$(a)\lfloor 2.3 \rfloor \quad (b)\lfloor 3 \rfloor \quad (c)\lfloor 4.31 \rfloor \quad (d)\lfloor 8.91 \rfloor$

$(e)\lfloor -0.7 \rfloor \quad (f)\lfloor -2.2 \rfloor \quad (g)\lfloor 7.999 \rfloor \quad (h)\lfloor 0 \rfloor$

2. Compute:

$(a)\lfloor \frac{20}{7} \rfloor \quad (b)\lfloor \frac{100}{15} \rfloor \quad (c)\lfloor \frac{26}{8} \rfloor$

$(d)\lfloor \frac{144}{5} \rfloor \quad (e)\lfloor \frac{27}{7} \rfloor \quad (f)\lfloor \frac{16}{3} \rfloor$

3. (a) List the positive integral multiples of 7 that are less than or equal to 20.

(b) List the positive integral multiples of 15 that are less than or equal to 100.

4. (a) Explain why $n \cdot \lfloor \frac{m}{n} \rfloor$ is the largest multiple of n less than or equal to m.

(b) If $p^{nk} | p^m$, show that $k \leq \lfloor \frac{m}{n} \rfloor$.

5. Explain why $\lfloor \frac{m}{n} \rfloor$ counts the number of positive integral multiples of n not greater than m.

6. How many positive integral multiples of 7 are between (a) 7 and 49; (b) 1 and 101 inclusive?

7. How many positive integral multiples of 11 are between (a) 49 and 196; (b) 81 and 281 inclusive?

Factorial: Define $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$.

8. List $n!$ for each $1 \leq n \leq 8$.

9. Compute $v_2(n)$ for each $3 \leq n \leq 8$.

10. Find:

$(a)\lfloor \frac{3}{2} \rfloor + \lfloor \frac{3}{4} \rfloor \quad (b)\lfloor \frac{6}{2} \rfloor + \lfloor \frac{6}{4} \rfloor + \lfloor \frac{6}{8} \rfloor \quad (c)\lfloor \frac{8}{2} \rfloor + \lfloor \frac{8}{4} \rfloor + \lfloor \frac{8}{8} \rfloor$

11. In this question, we want to find $v_2(16!)$.

(a) Write $16!$ as the product of 16 consecutive numbers.

(b) Under each number in part (a), put one dot if it is divisible by 2, two dots if it is divisible by 4, three dots if it is divisible by 8, and 4 dots if it is divisible by 16. There should be 4 rows of dots. Explain why $v_2(16!)$ equals the number of dots.

(c) Which numbers have dots under them? For these numbers, cross out one dot using an X. How many numbers have Xs below them?

(d) Which numbers have dots still not crossed out under them? For these numbers, cross out one dot using a Y. How many numbers have Ys under them?
(e) Explain why $v_2(16!)$ is the number of multiples of 2, plus the number of multiples of 4, plus the multiples of 8, plus the multiples of 16 less than or equal to 16.
(f) Find $v_2(16!)$.
12. Find $v_2(25!)$ and $v_5(100!)$.
13. Show that
$v_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots$.
14. (a) Explain why $v_2(2016!) = 1008 + 504 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1$, and evaluate it.
(b) Write a similar sum for $v_5(20160!)$ and evaluate it.
(c) How many zeroes does $2016!$ end in? Does this equal $v_2(2016!)$ or $v_5(2016!)$?

Phi function:
Define $\phi(n)$ to be the number of positive integers less than n and relatively prime to n. For example, $\phi(4) = 2$ because 1 and 3 are those positive integers.

15. Compute $\phi(6), \phi(8), \phi(10), \phi(12), \phi(14)$.
16. Compute $\phi(243)$ and $\phi(729)$.
17. Show that $\phi(p) = p - 1$ for any prime p.
18. Show that $\phi(p^k) = p^{k-1}(p-1)$ for any prime p.
19. Compute $\phi(35)$ and $\phi(5)\phi(7)$.
20. How many positive integers less than 1000 are multiples of 3?
21. How many positive integers less than 1000 are multiples of 2 or 3?
22. How many positive integers less than 1000 are multiples of 2, 3, or 5?
23. Compute $\phi(1000)$ and $\phi(3000)$.
24. Using principle of inclusion-exclusion on multiple variables, intuitively explain why
$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3}) \dots (1 - \frac{1}{p_k})$,
where $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ is the prime factorization of n.
25. How many three-digit integers are not divisible by 2, 3, or 5?